On Second-Order Error Terms In Fully Exponential Laplace Approximations.

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Abstract

Posterior means can be expressed as the ratio of integrals, which is called *fully exponential form.* To approximate the posterior means analytically, Laplace's method might be useful. In this article, we present explicit error terms of order n^{-1}, and of order n^{-2} in the Laplace approximations with asymptotic modes. Moreover, we give second-order error terms in fully exponential Laplace approximations to posterior means with asymptotic modes, which are proposed by Miyata (2004).

1. Introduction

Laplace's method for the asymptotic evaluation of integrals (Laplace, 1847) is a simple and useful technique. This method has been applied frequently in statistical theory by many authors (Mosteller and Wallace 1964; Lindley 1980; Tierney and Kadane 1986). $\Theta = \Theta_1 \times \cdots \times \Theta_d \subseteq \mathbb{R}^d$ is an open parameter space of $\boldsymbol{\theta} \equiv (\theta_1, ..., \theta_d)^T$, where ^{T} denotes the transpose of a matrix. Suppose that $(\Omega, \mathcal{A})$ is a measurable space, $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is a family of probability distributions on $(\Omega, \mathcal{A})$, and $\{\mathbf{X}_i : i = 1, 2, ...\}$ is a stochastic process on $(\Omega, \mathcal{A})$ with $\mathbf{X}_i$'s taking values in $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a subset of $\mathbb{R}$, and $\mathcal{B}$ is the class of Borel subsets of $\mathcal{X}$. We will assume that for all n, the distributions of $\tilde{\mathbf{X}} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ are dominated by a σ-finite measure, and we will denote a density of $\tilde{\mathbf{X}}$ under P_θ by $p_n(\mathbf{x}|\boldsymbol{\theta})$. Note that $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n, ...)$ is the observed sequence and $p_n(\mathbf{x}|\boldsymbol{\theta})$ depends on the first n observation $\tilde{\mathbf{x}} = (\mathbf{x}_1, ..., \mathbf{x}_n)$. Let θ_0 be a true parameter, and let "a.e. P_{θ_0}" be abbreviated to "a.e." or omitted. Let $g^+(\boldsymbol{\theta})$ be a smooth and strictly positive function. The purpose of this article is to give first-order and second-order error terms in the fully exponential Laplace approximation to the posterior mean

$$E[g^+(\boldsymbol{\Theta})] = \frac{\int_\Theta g^+(\boldsymbol{\theta}) p_n(\mathbf{x}|\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) d\boldsymbol{\theta}}{\int_\Theta p_n(\mathbf{x}|\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) d\boldsymbol{\theta}}, \tag{1.1}$$

where Θ is an open subset of $\mathbb{R}^d$, $p_n(\mathbf{x}|\boldsymbol{\theta})$ is the likelihood, and π is a prior. Although Θ also denotes a random vector with a posterior distribution, we can see from the context which Θ indicates. For convenience, the integrals in (1.1) are reexpressed as

$$E[g^+(\boldsymbol{\Theta})] = \frac{\int_\Theta exp\{-nh_n^*(\boldsymbol{\theta})\} d\boldsymbol{\theta}}{\int_\Theta exp\{-nh_n(\boldsymbol{\theta})\} d\boldsymbol{\theta}}, \tag{1.2}$$

where $h_n^*(\boldsymbol{\theta}) = -n^{-1} \log[g^+(\boldsymbol{\theta}) p_n(\mathbf{x}|\boldsymbol{\theta}) \pi(\boldsymbol{\theta})]$ and $h_n(\boldsymbol{\theta}) = -n^{-1} \log[p_n(\mathbf{x}|\boldsymbol{\theta}) \pi(\boldsymbol{\theta})]$.

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Section 2 sketches out the concept of asymptotic modes, and describes the Laplace approximations with asymptotic modes. In particular, the first-order and second-order errors are given in an explicit form. In Section 3, we give the explicit error terms of order n^{-2} in a fully exponential Laplace approximation to a posterior mean of g^+.

2. Asymptotic modes and Laplace's approximations

This section introduces the Laplace method for an integral of the form $\int_\Theta e^{-nh_n(\boldsymbol{\theta})}d\boldsymbol{\theta}$ with an asymptotic mode of $-h_n(\boldsymbol{\theta})$. For convenience of exposition, we write

$$\frac{\partial^s}{\partial\theta_{i1}\cdots\partial\theta_{is}}h_n(\boldsymbol{\theta}) \equiv h_{i1\cdots is}(\boldsymbol{\theta}), \qquad \sum_{i_1\cdots i_s} \equiv \sum_{i_1=1}^{d}\cdots\sum_{i_s=1}^{d},$$

and the first and second derivatives are represented by

$$D^1h_n(\boldsymbol{\theta}) \equiv \frac{\partial}{\partial\boldsymbol{\theta}}h_n(\boldsymbol{\theta}) = (\frac{\partial}{\partial\theta_1}h_n(\boldsymbol{\theta}),\cdots,\frac{\partial}{\partial\theta_d}h_n(\boldsymbol{\theta}))^T$$

$$D^2h_n(\boldsymbol{\theta}) \equiv \frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}h_n(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial^2}{\partial\theta_1\partial\theta_1}h_n(\boldsymbol{\theta}) & \cdots & \frac{\partial^2}{\partial\theta_1\partial\theta_d}h_n(\boldsymbol{\theta}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial\theta_d\partial\theta_1}h_n(\boldsymbol{\theta}) & \cdots & \frac{\partial^2}{\partial\theta_d\partial\theta_d}h_n(\boldsymbol{\theta}) \end{pmatrix}.$$

We say that $\hat{\boldsymbol{\theta}}$ is *an asymptotic mode of* $-h_n$ if $\hat{\boldsymbol{\theta}}$ converges to the exact mode of $-h_n(\boldsymbol{\theta})$ as the sample size n tends to infinity. The exact mode of $-h_n(\boldsymbol{\theta})$ is denoted by $\hat{\boldsymbol{\theta}}_{EX}$. Note that since $\hat{\boldsymbol{\theta}}_{EX}$ satisfies $D^1h_n(\hat{\boldsymbol{\theta}}_{EX}) = 0$, it is also taken as an asymptotic mode for $-h_n$. Additionally, the following asymptotic modes are defined.

Definition 1 $\hat{\boldsymbol{\theta}}$ is *an asymptotic mode of order* n^{-1} *for* $-h_n$ if $||\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{EX}|| \longrightarrow 0$ a.e., and $D^1h_n(\hat{\boldsymbol{\theta}}) = O(n^{-1})$ a.e.

Remark 1. Let $h_n(\boldsymbol{\theta}) = -n^{-1}\log[p_n(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})]$. The maximum likelihood estimator $\hat{\boldsymbol{\theta}}_{ML}$ for $p_n(\mathbf{x}|\boldsymbol{\theta})$ is an asymptotic mode of order n^{-1} for $-h_n$ because $D^1h_n(\hat{\boldsymbol{\theta}}_{ML}) = -n^{-1}\pi(\hat{\boldsymbol{\theta}}_{ML}) = O(n^{-1})$, and $\hat{\boldsymbol{\theta}}_{ML}$ converges to the exact mode of $-h_n(\boldsymbol{\theta})$, as n tends to infinity under suitable conditions. For the convergence of $\hat{\boldsymbol{\theta}}_{ML}$, see Heyde and Johnstone (1979).

Definition 2 $\hat{\boldsymbol{\theta}}$ is *an asymptotic mode of order* n^{-2} *for* $-h_n$ if $||\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{EX}|| \longrightarrow 0$ a.e., and $D^1h_n(\hat{\boldsymbol{\theta}}) = O(n^{-2})$ a.e.

Remark 2. Let $\hat{\boldsymbol{\theta}}_{ML}$ be the maximum likelihood estimator for $p(\mathbf{x}|\boldsymbol{\theta})$, and let $h_n(\boldsymbol{\theta}) = -n^{-1}\log[p(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})]$. Then, it follows from the same argument as in the proof of Theorem 3 of Miyata (2004) that $\hat{\boldsymbol{\theta}}^*_{ML} \equiv \hat{\boldsymbol{\theta}}_{ML} - [D^2h_n(\hat{\boldsymbol{\theta}}_{ML})]^{-1}D^1h_n(\hat{\boldsymbol{\theta}}_{ML})$ satisfies $D^1h_n(\hat{\boldsymbol{\theta}}^*_{ML}) = O(n^{-2})$.

Subsequently, we introduce the regularity conditions (A1), (A2), (A3), (A4) and (A5) for which the asymptotic expansions for $\int_\Theta e^{-nh_n(\boldsymbol{\theta})}d\boldsymbol{\theta}$ will be valid. Let $||\mathbf{a}|| \equiv (\mathbf{a}^T\mathbf{a})^{1/2}$ for any vector

$\mathbf{a}$, $|\cdot|$ denote the determinant of a matrix. We use $B_\delta(\hat{\boldsymbol{\theta}})$ to denote the open ball of radius δ centered at $\hat{\boldsymbol{\theta}}$, namely $B_\delta(\hat{\boldsymbol{\theta}}) = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} : ||\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}|| < \delta\}$. Let $\{\hat{\boldsymbol{\theta}}\} \equiv \{\hat{\boldsymbol{\theta}} : n = 1, 2, ...\}$ be the sequence of asymptotic modes.

We list the following assumptions for $(\{h_n(\boldsymbol{\theta})\}, \{\hat{\boldsymbol{\theta}}\})$:

(A1) $\{h_n(\boldsymbol{\theta}) : n = 1, 2, ...\}$ is a sequence of eight times continuously differentiable real functions on $\boldsymbol{\Theta}$.

There exists positive numbers ϵ, M, ζ and an integer n_0 such that for the asymptotic mode $\hat{\boldsymbol{\theta}}$, $n \geq n_0$ implies

(A2) the integral $\int_\Theta e^{-nh_n(\boldsymbol{\theta})} d\boldsymbol{\theta}$ is finite;

(A3) for all $\boldsymbol{\theta} \in B_\epsilon(\hat{\boldsymbol{\theta}})$ and all $1 \leq j_1, ..., j_m \leq d$ with $m = 1, ..., 8$, $||h_n(\boldsymbol{\theta})|| < M$ and $||\partial^m h_n(\boldsymbol{\theta})/\partial\theta_{j_1} \cdots \partial\theta_{j_m}|| < M$;

(A4) $D^2 h_n(\hat{\boldsymbol{\theta}})$ is positive definite and $|D^2 h_n(\hat{\boldsymbol{\theta}})| > \zeta$;

(A5) for all δ for which $0 < \delta < \epsilon$, $B_\delta(\hat{\boldsymbol{\theta}}) \subseteq \boldsymbol{\Theta}$ and

$$|nD^2 h_n(\hat{\boldsymbol{\theta}})|^{1/2} C_n(\hat{\boldsymbol{\theta}})^{-1} \int_{\boldsymbol{\Theta} - B_\delta(\hat{\boldsymbol{\theta}})} exp\{-n[h_n(\boldsymbol{\theta}) - h_n(\hat{\boldsymbol{\theta}})]\} d\boldsymbol{\theta} = O(n^{-3}),$$

where $C_n(\hat{\boldsymbol{\theta}}) = exp\left(\frac{n}{2} D^1 h(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 h(\hat{\boldsymbol{\theta}})\right)$. The pair $(\{h_n\}, \{\hat{\boldsymbol{\theta}}\})$ will be said to satisfy the *analytical assumptions for the asymptotic-mode Laplace method* if (A1), (A2), (A3), (A4) and (A5) are satisfied for the asymptotic mode $\hat{\boldsymbol{\theta}}$.

Our conditions (A1)–(A5) are analogous to those of Kass, Tierney and Kadane (1990) except $\hat{\boldsymbol{\theta}}$ is an asymptotic mode. If $\hat{\boldsymbol{\theta}}$ is an asymptotic mode of order n^{-1}, $\int_{\boldsymbol{\Theta} - B_\delta(\hat{\boldsymbol{\theta}})} exp\{-n[h_n(\boldsymbol{\theta}) - h_n(\hat{\boldsymbol{\theta}})]\} d\boldsymbol{\theta} = O(n^{-3-d/2})$ holds under (A5). Hence (A5) means that the probability outside a neighborhood of the $\hat{\boldsymbol{\theta}}$ converges to zero as the sample size n tends to infinity. Let $h_{j1 \cdots jm}$ denote the mth partial derivative $\partial^m h_n(\boldsymbol{\theta})/\partial\theta_{j1} \cdots \partial\theta_{jm}$ with respect to $\boldsymbol{\theta}$ evaluated at $\hat{\boldsymbol{\theta}}$, for example, h_{112} means $\partial^3 h_n(\hat{\boldsymbol{\theta}})/\partial\hat{\theta}_1^2 \partial\hat{\theta}_2$. Let h^{ij} be the components of $[D^2 h(\hat{\boldsymbol{\theta}})]^{-1}$ and $\mathbf{b} = (b_i) \equiv -[D^2 h_n(\hat{\boldsymbol{\theta}})]^{-1} D^1 h_n(\hat{\boldsymbol{\theta}})$. We define the sixth, eighth, tenth, and twelfth central moments of a multivariate normal distribution having covariance matrix $[nD^2 h(\hat{\boldsymbol{\theta}})]^{-1}$ as μ_{ijkqrs}, $\mu_{ijkqrstu}$, $\mu_{ijkqrstuvw}$, and $\mu_{ijkqrstuvwxy}$, respectively.

Theorem 3. Suppose that $\hat{\boldsymbol{\alpha}}$ is an asymptotic mode of order n^{-1} for $-h_n$ and the pair $(\{h_n(\boldsymbol{\alpha})\}, \{\hat{\boldsymbol{\alpha}}\})$ satisfies *analytical assumptions for the asymptotic-mode Laplace method.* Then it follows that for large n,

$$\int_\Theta e^{-nh_n(\boldsymbol{\theta})} d\boldsymbol{\theta} = (2\pi)^{d/2} |\Sigma|^{1/2} e^{-nh_n(\hat{\boldsymbol{\alpha}})} C_n(\hat{\boldsymbol{\alpha}}) \left(1 + \frac{\lambda_{1n}}{n} + \frac{\lambda_{2n}}{n^2} + O(n^{-3})\right), \tag{2.1}$$

where $C_n(\hat{\boldsymbol{\alpha}}) = exp\left(\frac{n}{2} D^1 h_n(\hat{\boldsymbol{\alpha}})^T [D^2 h_n(\hat{\boldsymbol{\alpha}})]^{-1} D^1 h_n(\hat{\boldsymbol{\alpha}})\right)$,

$$\lambda_{1n} = -\frac{1}{2}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\alpha}})(nb_i)h^{jk} - \frac{1}{8}\sum_{ijkq} h_{ijkq}(\hat{\boldsymbol{\alpha}})h^{ij}h^{kq}$$
$$+ \frac{1}{72}\sum_{ijkqrs} h_{ijk}(\hat{\boldsymbol{\alpha}})h_{qrs}(\hat{\boldsymbol{\alpha}})\mu_{ijkqrs}n^3,$$

$$\begin{aligned}
\lambda_{2n} = &-\frac{n^3}{6}\sum h_{ijk}b_ib_jb_k - \frac{n^2}{4}\sum h_{ijkq}h^{ij}b_kb_q + \frac{n^4}{12}\sum h_{ijk}h_{qrs}b_ib_j\mu_{kqrs} \\
&+ \frac{n^4}{8}\sum h_{ijk}h_{qrs}b_ib_q\mu_{jkrs} - \frac{n^3}{24}\sum h_{ijkqr}b_i\mu_{jkqr} \\
&- \frac{n^3}{720}\sum h_{ijkqrs}\mu_{ijkqrs} + \frac{n^4}{48}\sum h_{ijk}h_{qrst}b_i\mu_{jkqrst} \\
&+ \frac{n^4}{36}\sum h_{ijk}h_{qrst}b_q\mu_{ijkrst} + \frac{n^4}{1152}\sum h_{ijkq}h_{rstu}\mu_{ijkqrstu} \\
&+ \frac{n^4}{720}\sum h_{ijk}h_{qrstu}\mu_{ijkqrstu} - \frac{n^5}{144}\sum h_{ijk}h_{qrs}h_{tuv}b_i\mu_{jkqrstuv} \\
&- \frac{n^5}{1728}\sum h_{ijk}h_{qrs}h_{tuvw}\mu_{ijkqrstuvw} \\
&+ \frac{n^6}{31104}\sum h_{ijk}h_{qrs}h_{tuv}h_{wxy}\mu_{ijkqrstuvwxy},
\end{aligned} \tag{2.2}$$

$\Sigma \equiv [nD^2h(\hat{\boldsymbol{\alpha}})]^{-1} = (n^{-1}h^{ij})$, and μ_{ijkqrs} are the sixth central moments of a multivariate Normal distribution having covariance matrix Σ.

Note that λ_{1n} and λ_{2n} are of order $O(1)$. The proof is given in Appendix C.

Theorem 4. Suppose that $\hat{\boldsymbol{\theta}}$ is an asymptotic mode of order n^{-2} for $-h_n$ and the pair $(\{h_n(\boldsymbol{\theta})\}, \{\hat{\boldsymbol{\theta}}\})$ satisfies *analytical assumptions for the asymptotic-mode Laplace method.* Then it follows that for large n,

$$\int_{\Theta} e^{-nh_n(\boldsymbol{\theta})}d\boldsymbol{\theta} = (2\pi)^{d/2}|\Sigma|^{1/2}e^{-nh_n(\hat{\boldsymbol{\theta}})}C_n(\hat{\boldsymbol{\theta}})\left(1 + \frac{a_{1n}}{n} + \frac{a_{2n}}{n^2} + O(n^{-3})\right), \tag{2.3}$$

where $C_n(\hat{\boldsymbol{\theta}}) = exp\left(\frac{n}{2}D^1h_n(\hat{\boldsymbol{\theta}})^T[D^2h_n(\hat{\boldsymbol{\theta}})]^{-1}D^1h_n(\hat{\boldsymbol{\theta}})\right)$,

$$\begin{aligned}
a_{1n} = &-\frac{1}{8}\sum_{ijkq} h_{ijkq}h^{ij}h^{kq} + \frac{n^3}{72}\sum_{ijkqrs} h_{ijk}h_{qrs}\mu_{ijkqrs}, \\
a_{2n} = &-\frac{1}{2}\sum_{ijk} h_{ijk}(n^2b_i)h^{jk} - \frac{n^3}{720}\sum h_{ijkqrs}\mu_{ijkqrs} \\
&+ \frac{n^4}{1152}\sum h_{ijkq}h_{rstu}\mu_{ijkqrstu} + \frac{n^4}{720}\sum h_{ijk}h_{qrstu}\mu_{ijkqrstu} \\
&- \frac{n^5}{1728}\sum h_{ijk}h_{qrs}h_{tuvw}\mu_{ijkqrstuvw} \\
&+ \frac{n^6}{31104}\sum h_{ijk}h_{qrs}h_{tuv}h_{xyz}\mu_{ijkqrstuvwxy},
\end{aligned}$$

and $\Sigma \equiv [nD^2h(\hat{\boldsymbol{\theta}})]^{-1} = (n^{-1}h^{ij})$.

a_{1n}, and a_{2n} are of order $O(1)$, and the proof is given in Appendix C.

Remark 5. *$O(n^{-3})$ includes $h_i, ..., h_{ijkqrst}$, h^{ij}, b_i,*

$$\frac{-n}{8!}\int_{B_\delta(\hat{\boldsymbol{\theta}})} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)h_{ijkqrstu}(\gamma_1)z_iz_jz_kz_qz_rz_sz_tz_u d\boldsymbol{\theta},$$
$$\int_{\boldsymbol{\Theta}-B_\delta(\hat{\boldsymbol{\theta}})} exp\{-n[h_n(\boldsymbol{\theta})-h_n(\hat{\boldsymbol{\theta}})]\}d\boldsymbol{\theta}$$
$$and \int_{B_\delta(\hat{\boldsymbol{\theta}})} n(\boldsymbol{\theta}|\hat{\boldsymbol{\theta}}+\mathbf{b},[nD^2h_n(\hat{\boldsymbol{\theta}})]^{-1})R_{5n}(k_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}))d\boldsymbol{\theta},$$

where γ_1 is a point between $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$,

$$k_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) = -n\Big(\frac{1}{6}\sum_{ijk}h_{ijk}z_iz_jz_k+\cdots+\frac{1}{7!}\sum_{ijkqrst}h_{ijkqrst}z_i\cdots z_t+\frac{1}{8!}\sum_{ijkqrstu}h_{ijkqrstu}(\gamma_2)z_i\cdots z_u\Big)$$

γ_2 is a point between $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$

$$R_{5n}(k_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})) = \frac{k_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})^5}{4!}\int_0^1(1-\lambda)^4 exp\{\lambda k_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})\}d\lambda.$$

3. Main results

This section mainly gives the explicit error terms of order n^{-2} in the fully exponential Laplace approximations to posterior means. Let $\mathbf{b}^* = (b_i^*) \equiv -[D^2h_n^*(\hat{\boldsymbol{\theta}})]^{-1}D^1h_n^*(\hat{\boldsymbol{\theta}})$

Theorem 6. Let $\hat{\boldsymbol{\theta}}$ be an asymptotic mode of order n^{-2} for $-h_n(\boldsymbol{\theta})$, and $\hat{\boldsymbol{\theta}}_N$ is a single Newton step from $\hat{\boldsymbol{\theta}}$ toward the maximum of $-h_n^*(\boldsymbol{\theta})$, i.e., $\hat{\boldsymbol{\theta}}_N \equiv \hat{\boldsymbol{\theta}} - [D^2h_n^*(\hat{\boldsymbol{\theta}})]^{-1}D^1h_n^*(\hat{\boldsymbol{\theta}})$. If pairs $(\{h_n(\boldsymbol{\theta})\},\{\hat{\boldsymbol{\theta}}\})$ and $(\{h_n^*(\boldsymbol{\theta})\},\{\hat{\boldsymbol{\theta}}_N\})$ satisfy *analytical assumptions for the asymptotic-mode Laplace method*, Then

$$E[g(\boldsymbol{\Theta})] = \Big(\frac{|D^2h_n(\hat{\boldsymbol{\theta}})|}{|D^2h_n^*(\hat{\boldsymbol{\theta}}_N)|}\Big)^{1/2}\frac{C_n^*(\hat{\boldsymbol{\theta}}_N)}{C_n(\hat{\boldsymbol{\theta}})}\frac{exp[-nh^*(\hat{\boldsymbol{\theta}}_N)]}{exp[-nh(\hat{\boldsymbol{\theta}})]}\Big(1+\frac{c_n}{n^2}+O(n^{-3})\Big), \tag{3.1}$$

where $C_n^*(\hat{\boldsymbol{\theta}}_N) = exp\{(n/2)D^1h_n^*(\hat{\boldsymbol{\theta}}_N)^T[D^2h_n^*(\hat{\boldsymbol{\theta}}_N)]^{-1}D^1h_n^*(\hat{\boldsymbol{\theta}}_N)\}$ and

$$\begin{aligned}
c_n = &-\frac{n^2}{2}\sum h_{ijk}b_i^*h^{jk}+\frac{n^2}{2}\sum h_{ijk}b_ih^{jk}+\frac{1}{4}\sum h_{ijkq}h_{l\alpha\beta}h^{ij}h^{k\alpha}h^{\beta q}h^{lm}G_m\\
&-\frac{1}{4}\sum h_{ijkq}h^{ij}h^{k\alpha}h^{\beta q}G_{\alpha\beta}-\frac{1}{8}\sum h_{ijkql}h^{ij}h^{kq}h^{lm}G_m+\frac{1}{8}\sum h^{ij}h^{kq}G_{ijkq}\\
&+\frac{1}{36}\sum h_{\alpha ijk}h_{qrs}h^{\alpha\beta}[n^3\mu_{ijkqrs}]G_\beta-\frac{1}{36}\sum h_{qrs}[n^3\mu_{ijkqrs}]G_{ijk}\\
&-\frac{1}{8}\sum_{ijkqrs\alpha\beta lm}h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}h^{lm}G_m+\frac{1}{8}\sum_{ijkqrs\alpha\beta}h_{ijk}h_{qrs}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}G_{\alpha\beta}\\
&-\frac{1}{4}\sum_{ijkqrs\alpha\beta lm}h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}h^{lm}G_m+\frac{1}{4}\sum_{ijkqrs\alpha\beta}h_{ijk}h_{qrs}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}G_{\alpha\beta}\\
&-\frac{1}{4}\sum_{ijkqrs\alpha\beta lm}h_{ijk}h_{qrs}h_{l\alpha\beta}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}h^{lm}G_m+\frac{1}{4}\sum_{ijkqrs\alpha\beta}h_{ijk}h_{qrs}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}G_{\alpha\beta}.
\end{aligned}$$

Note that c_n is of order $O(1)$. The first term in the right side of c_n is reexpressed as

$$-\frac{n^2}{2}\sum h_{ijk}b_i^* h^{jk} = \frac{1}{4}\sum_{ijklmr\alpha\beta} h_{ijk}h_{l\alpha\beta}h^{il}h^{jk}h^{\alpha r}h^{\beta m}G_rG_m + O(n^{-1}). \tag{3.2}$$

If $\hat{\boldsymbol{\theta}}$ is replaced with the exact mode $\hat{\boldsymbol{\theta}}_{EX}$ of $-h_n(\boldsymbol{\theta})$ in (3.1), then it follows that

$$E[g(\boldsymbol{\Theta})] = \left(\frac{|D^2h_n(\hat{\boldsymbol{\theta}}_{EX})|}{|D^2h_n^*(\hat{\boldsymbol{\theta}}_N)|}\right)^{1/2} C_n^*(\hat{\boldsymbol{\theta}}_N)\frac{exp[-nh^*(\hat{\boldsymbol{\theta}}_N)]}{exp[-nh(\hat{\boldsymbol{\theta}}_{EX})]}\left(1+\frac{\dot{c}_n}{n^2}+O(n^{-3})\right), \tag{3.3}$$

where $\dot{c}_n$ is c_n with the term $(n^2/2)\sum h_{ijk}b_ih^{jk}$ being 0. Furthermore, the approximation with $\hat{\boldsymbol{\theta}}_N$ with the exact mode of $-h_n^*$ is equivalent to the Tierney–Kadane approximation. As a generalized form, the following result holds.

Theorem 7. Suppose that $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ are asymptotic modes of order n^{-2} for $-h_n(\boldsymbol{\theta})$ and $-h_n^*(\boldsymbol{\theta})$ respectively. Then

$$E[g(\boldsymbol{\Theta})] = \left(\frac{|D^2h_n(\hat{\boldsymbol{\theta}})|}{|D^2h_n^*(\tilde{\boldsymbol{\theta}})|}\right)^{1/2} \frac{C_n^*(\tilde{\boldsymbol{\theta}})}{C_n(\hat{\boldsymbol{\theta}})}\frac{exp[-nh^*(\tilde{\boldsymbol{\theta}})]}{exp[-nh(\hat{\boldsymbol{\theta}})]}\left(1+O(n^{-2})\right). \tag{3.4}$$

Appendix A: Lemmas concerning matrices

This section prepares some lemmas concerning matrices to prove the main results.

Lemma 8. Suppose that $\mathbf{a}$ is a $1\times d$ matrix, $\mathbf{M}$ is a $d\times d$ symmetric matrix. Then,

$$\mathbf{a}^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\mathbf{M}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) = (\boldsymbol{\theta}-\mathbf{y})^T\mathbf{M}(\boldsymbol{\theta}-\mathbf{y}) - \frac{1}{4}\mathbf{a}^T\mathbf{M}^{-1}\mathbf{a},$$

where $\mathbf{y} = \hat{\boldsymbol{\theta}} - (1/2)\mathbf{M}^{-1}\mathbf{a}$.

Proof.

$$\begin{aligned}
&(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}+\frac{1}{2}\mathbf{M}^{-1}\mathbf{a})^T\mathbf{M}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}+\frac{1}{2}\mathbf{M}^{-1}\mathbf{a}) - \frac{1}{4}\mathbf{a}^T M^{-1}\mathbf{a}\\
&= (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\mathbf{M}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \frac{1}{2}\mathbf{a}^T\mathbf{M}^{-1}\mathbf{M}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\mathbf{M}\frac{1}{2}\mathbf{M}^{-1}\mathbf{a} + \frac{1}{2}\mathbf{a}^T\mathbf{M}^{-1}\mathbf{M}\frac{1}{2}\mathbf{M}^{-1}\mathbf{a}\\
&\qquad - \frac{1}{4}\mathbf{a}^T\mathbf{M}^{-1}\mathbf{a}\\
&= \mathbf{a}^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\mathbf{M}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}).
\end{aligned}$$

Lemma 9. Let $\mathbf{I}_d$ be a $d\times d$ unit matrix. For any $d\times d$ matrix $\mathbf{A}$ and any scalar x,

$$|\mathbf{A}+x\mathbf{I}_d| = \sum_{r=0}^{d} x^r \sum_{\{i_1,\dots,i_r\}} |\mathbf{A}^{\{i_1,\dots,i_r\}}|, \tag{A.1}$$

where $|\cdot|$ denotes the determinant of a matrix, $\{i_1,...,i_r\}$ is an r-dimensional subset of the first d positive integers $1,...,d$ (and the second summation is over all $\binom{d}{r}$ such subsets), and $\mathbf{A}^{\{i_1,\dots,i_r\}}$ is the $(d-r)\times(d-r)$ principal submatrix of A obtained by striking out the $i_1,...,i_r$th rows and columns. (For $r=d$, the sum $\sum_{\{i_1,\dots,i_r\}}|\mathbf{A}^{\{i_1,\dots,i_r\}}|$ is to be interpreted as 1.) Note that the expression (A.1) is a polynomial in x, the coefficient of x^0 (i.e., the constant term of the polynomial) equals $|\mathbf{A}|$, and the coefficient of x^{d-1} equals $tr(\mathbf{A})$.

Proof. See Harville (1997; corollary 13.7.4, p.197).

Lemma 10. For $d \times d$ nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\frac{d}{dt}\left(\frac{|\mathbf{A}-(t/n)\mathbf{B}|}{|\mathbf{A}|}\right)\Big|_{t=0} = -\frac{1}{n}tr(\mathbf{B}\mathbf{A}^{-1}).$$

Proof. Using Lemma 8 and the relation between an inverse matrix and the cofactors, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{|\mathbf{A}-(t/n)\mathbf{B}|}{|\mathbf{A}|}\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(|\mathbf{B}\mathbf{A}^{-1}||\mathbf{A}\mathbf{B}^{-1}-\frac{t}{n}\mathbf{I}_d|\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(|\mathbf{B}\mathbf{A}^{-1}|\left[|\mathbf{A}\mathbf{B}^{-1}| - \frac{t}{n}\sum_{\{i_1\}}|(\mathbf{A}\mathbf{B}^{-1})^{\{i_1\}}| + \frac{t^2}{n^2}\xi + \cdots + \frac{(-t)^d}{n^d}\right]\right)\Big|_{t=0} \\
&= -\frac{1}{n}\frac{1}{|\mathbf{A}\mathbf{B}^{-1}|}\sum_{\{i_1\}}|(\mathbf{A}\mathbf{B}^{-1})^{\{i_1\}}| \\
&= -\frac{1}{n}tr(\mathbf{A}\mathbf{B}^{-1})^{-1} \\
&= -\frac{1}{n}tr(\mathbf{B}\mathbf{A}^{-1}),
\end{aligned}$$

where $\xi = \sum_{\{i_1,i_2\}}|(\mathbf{A}\mathbf{B}^{-1})^{\{i_1,i_2\}}|$ and $\{i_1\}$ and $\{i_1, i_2\}$ are the same notations as in Lemma 8. Hence this lemma is proved.

Lemma 11. For $d \times d$ nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$ and any scalar x, it follows that

$$(\mathbf{A}+x\mathbf{B})^{-1} - \mathbf{A}^{-1} = O(x). \tag{A.2}$$

Proof. Multiplying $(\mathbf{A}+x\mathbf{B})^{-1}(\mathbf{A}+x\mathbf{B}) = \mathbf{I}_d$ by $\mathbf{A}^{-1}$ from the right side yields

$$(\mathbf{A}+x\mathbf{B})^{-1} - \mathbf{A}^{-1} = -x(\mathbf{A}+x\mathbf{B})^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{A.3}$$

Hence the lemma is proved.

Lemma 12. For $d \times d$ nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\frac{d}{dt}(\mathbf{A}-\frac{t}{n}\mathbf{B})^{-1}\Big|_{t=0} = \frac{1}{n}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}.$$

Proof. Using (A.3),

$$\frac{(\mathbf{A}-(t/n)\mathbf{B})^{-1}-\mathbf{A}^{-1}}{t} = \frac{1}{n}(\mathbf{A}-\frac{t}{n}\mathbf{B})^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{A.4}$$

Hence, letting $t \to 0$ yields the lemma.

Appendix B: Some lemmas

This section presents some lemmas required in the evaluation of the asymptotic errors in Sections 2 and 3.

Lemma 13. Let n denote a sample size, and let $z_i \equiv \Theta_i - \hat{\theta}_i$. Suppose that $\mathbf{\Theta} = (\Theta_1, ..., \Theta_d)^T$ is according to $N_d(\hat{\boldsymbol{\theta}} + \mathbf{b}, \Sigma)$, where $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, ..., \hat{\theta}_d)^T$, $\mathbf{b} = (b_1, ..., b_d)^T = O(n^{-1})$, and $\Sigma = (\mu_{ij})$ is the covariance matrix with $\mu_{ij} = O(n^{-1})$. Then we have the following results.

(a) $E[(\Theta_i - \hat{\theta}_i)(\Theta_j - \hat{\theta}_j)(\Theta_k - \hat{\theta}_k)] = \underbrace{b_i\mu_{jk} + b_j\mu_{ik} + b_k\mu_{ij}}_{O(n^{-2})} + \underbrace{b_ib_jb_k}_{O(n^{-3})}$

(b) $$E[(\Theta_i - \hat{\theta}_i)(\Theta_j - \hat{\theta}_j)(\Theta_k - \hat{\theta}_k)(\Theta_q - \hat{\theta}_q)] = \underbrace{\mu_{ij}\mu_{kq} + \mu_{ik}\mu_{jq} + \mu_{iq}\mu_{kj}}_{O(n^{-2})} + \sum_{[m_1,m_2]_4} \underbrace{b_{m_1}b_{m_2}\mu_{m_3m_4}}_{O(n^{-3})} + b_ib_jb_kb_q,$$

where $[m_1, m_2]_4$ indicates ways of a pair chosen among four alphabets i, j, k, and q. Hence the summation $\sum_{[m1,m2]_4}$ is over $\binom{4}{2}$.

(c) $$E[z_iz_jz_kz_qz_r] = \sum_{[m_1]_5} \underbrace{b_{m_1}\mu_{m_2m_3m_4m_5}}_{O(n^{-3})} + \sum_{[l_1,l_2,l_3]_5} \underbrace{b_{l_1}b_{l_2}b_{l_3}\mu_{l_4l_5}}_{O(n^{-4})} + b_ib_jb_kb_qb_r,$$

where $[m_1]_5$ indicates ways of one chosen among five alphabets i, j, k, q, and r, and $[l_1, l_2, l_3]_5$ indicates ways of a triple chosen among i, j, k, q, and r. The notations $[l_1, l_2, l_3, l_4]_6$, $[m_1, m_2]_6$, etc below are also defined similarly.

(d) $$E[z_iz_jz_kz_qz_rz_s] = \underbrace{\mu_{ijkqrs}}_{O(n^{-3})} + \sum_{[m_1,m_2]_6} \underbrace{b_{m_1}b_{m_2}\mu_{m_3m_4m_5m_6}}_{O(n^{-4})} + \sum_{[l_1,l_2,l_3,l_4]_6} \underbrace{b_{l_1}b_{l_2}b_{l_3}b_{l_4}\mu_{l_5l_6}}_{O(n^{-5})} + b_ib_jb_kb_qb_rb_s$$

(e) $$E[z_iz_jz_kz_qz_rz_sz_t] = \sum_{[m_1]_7} \underbrace{b_{m_1}\mu_{m_2m_3m_4m_5m_6m_7}}_{O(n^{-4})} + \sum_{[l_1,l_2,l_3]_7} b_{l_1}b_{l_2}b_{l_3}\mu_{l_4l_5l_6l_7} + \sum_{[n_1,n_2,n_3,n_4,n_5]_7} b_{n_1}b_{n_2}b_{n_3}b_{n_4}b_{n_5}\mu_{n_6n_7} + b_ib_jb_kb_qb_rb_sb_t$$

(f) $E[\underbrace{z_iz_jz_kz_qz_rz_sz_tz_u}_{8}] = \underbrace{\mu_{ijkqrstu}}_{O(n^{-4})} + O(n^{-5})$

(g) $E[\underbrace{z_iz_jz_kz_qz_rz_sz_tz_uz_v}_{9}] = \sum_{[m_1]_9} \underbrace{b_{m_1}\mu_{m_2m_3m_4m_5m_6m_7m_8m_9}}_{O(n^{-5})} + O(n^{-6})$

(h) $E[\underbrace{z_iz_jz_kz_qz_rz_sz_tz_uz_vz_w}_{10}] = \underbrace{\mu_{ijkqrstuvw}}_{O(n^{-5})} + O(n^{-6})$

(i) $E[\underbrace{z_iz_jz_kz_qz_rz_sz_tz_uz_vz_wz_xz_y}_{12}] = \underbrace{\mu_{ijkqrstuvwxy}}_{O(n^{-6})} + O(n^{-7})$

Proof of (a). Putting $u_i \equiv \Theta_i - (\hat{\theta}_i + b_i)$, $E[u_i] = 0$, and $E[u_i u_j] = \mu_{ij}$. By using the binomial theorem and the result in which odd moments of u_i are 0, we have

$$\begin{aligned}
&E[(\Theta_i - \hat{\theta}_i)(\Theta_j - \hat{\theta}_j)(\Theta_k - \hat{\theta}_k)] \\
&= E\Big[\big(\Theta_i - (\hat{\theta}_i + b_i) + b_i\big)\big(\Theta_j - (\hat{\theta}_j + b_j) + b_j\big)\big(\Theta_k - (\hat{\theta}_k + b_k) + b_k\big)\Big] \\
&= E[(u_i + b_i)(u_j + b_j)(u_k + b_k)] \\
&= E[u_i u_j u_k] + b_i E[u_j u_k] + b_j E[u_i u_k] + b_k E[u_i u_j] \\
&\quad + b_i b_j E[u_k] + b_j b_k E[u_i] + b_i b_k E[u_j] + b_i b_j b_k \\
&= b_i \mu_{jk} + b_j \mu_{ik} + b_k \mu_{ij} + b_i b_j b_k.
\end{aligned}$$

Proof of (b).

$$\begin{aligned}
&E[(\Theta_i - \hat{\theta}_i)(\Theta_j - \hat{\theta}_j)(\Theta_k - \hat{\theta}_k)(\Theta_q - \hat{\theta}_q)] \\
&= E\Big[\big(\Theta_i - (\hat{\theta}_i + b_i) + b_i\big)\big(\Theta_j - (\hat{\theta}_j + b_j) + b_j\big)\big(\Theta_k - (\hat{\theta}_k + b_k) + b_k\big)\big(\Theta_q - (\hat{\theta}_q + b_q) + b_q\big)\Big] \\
&= E[(u_i + b_i)(u_j + b_j)(u_k + b_k)(u_q + b_q)] \\
&= E[u_i u_j u_k u_q] + \sum_{[m_1]_4} b_{m_1} \underbrace{E[u_{m_2} u_{m_3} u_{m_4}]}_{0} + \sum_{[m_1, m_2]_4} b_{m_1} b_{m_2} E[u_{m_3} u_{m_4}] \\
&\quad + \sum_{[m_1, m_2, m_3]_4} b_{m_1} b_{m_2} b_{m_3} \underbrace{E[u_{m_4}]}_{0} + b_i b_j b_k b_q \\
&= \mu_{ijkq} + \sum_{[m_1, m_2]_4} b_{m_1} b_{m_2} E[u_{m_3} u_{m_4}] + b_i b_j b_k b_q.
\end{aligned}$$

Proof of (c).

$$\begin{aligned}
&E[z_i z_j z_k z_q z_r] \\
&= E[(\Theta_i - (\hat{\theta}_i + b_i) + b_i)(\Theta_j - (\hat{\theta}_j + b_j) + b_j) \cdots (\Theta_r - (\hat{\theta}_r + b_r) + b_r)] \\
&= E[(u_i + b_i)(u_j + b_j)(u_k + b_k)(u_q + b_q)(u_r + b_r)] \\
&= \underbrace{\mu_{ijkqr}}_{0} + \sum_{[m_1]_5} b_{m_1} E[u_{m_2} u_{m_3} u_{m_4} u_{m_5}] + \sum_{[m_1, m_2]_5} b_{m_1} b_{m_2} \underbrace{E[u_{m_3} u_{m_4} u_{m_5}]}_{0} \\
&\quad + \sum_{[m_1, m_2, m_3]_5} b_{m_1} b_{m_2} b_{m_3} E[u_{m_4} u_{m_5}] + \sum_{[m_1, m_2, m_3, m_4]_5} b_{m_1} b_{m_2} b_{m_3} b_{m_4} E[u_{m_5}] + b_i b_j b_k b_q b_r \\
&= \sum_{[m_1]_5} b_{m_1} \mu_{m_2 m_3 m_4 m_5} + \sum_{[m_1, m_2, m_3]_5} b_{m_1} b_{m_2} b_{m_3} \mu_{m_4 m_5} + b_i b_j b_k b_q b_r.
\end{aligned}$$

Proof of (d).

$$
\begin{aligned}
&E[z_i z_j z_k z_q z_r z_s] \\
&= E[(u_i + b_i)(u_j + b_j)(u_k + b_k)(u_q + b_q)(u_r + b_r)(u_s + b_s)] \\
&= \mu_{ijkqrs} + \sum_{[m_1]_6} b_{m_1} \underbrace{E[u_{m_2} u_{m_3} u_{m_4} u_{m_5} u_{m_6}]}_{0} \\
&\quad + \sum_{[m_1,m_2]_6} b_{m_1} b_{m_2} E[u_{m_3} u_{m_4} u_{m_5} u_{m_6}] + \sum_{[m_1,m_2,m_3]_6} b_{m_1} b_{m_2} b_{m_3} E[u_{m_4} u_{m_5} u_{m_6}] \\
&\quad + \sum_{[m_1,m_2,m_3,m_4]_6} b_{m_1} b_{m_2} b_{m_3} b_{m_4} E[u_{m_5} u_{m_6}] + \sum_{[m_1,m_2,m_3,m_4,m_5]_6} b_{m_1} b_{m_2} b_{m_3} b_{m_4} b_{m_5} E[u_{m_6}] \\
&\quad + b_i b_j b_k b_q b_r b_s.
\end{aligned}
$$

Similarly, (e)–(h) can be proved.

Let $G(\boldsymbol{\theta}) \equiv \log g^+(\boldsymbol{\theta})$ and let $G_{j_1 \ldots j_m}$ denote the mth partial derivatives $\partial^m G(\boldsymbol{\theta})/\partial\theta_{j_1}\ldots\theta_{j_m}$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. For example, $G_{213} = \partial^3 G(\hat{\boldsymbol{\theta}})/\partial\hat{\theta}_2 \partial\hat{\theta}_1 \partial\hat{\theta}_3$. For simplicity, we let $h_n^* = h^*$ and $h_n = h$.

Lemma 14. Let $\hat{\boldsymbol{\theta}}$ be an asymptotic mode of order n^{-2} for $-h(\boldsymbol{\theta})$, and let $h^*(\boldsymbol{\theta}) \equiv h(\boldsymbol{\theta}) - (1/n)G(\boldsymbol{\theta})$. Suppose that $h^*(\boldsymbol{\theta})$ and $h(\boldsymbol{\theta})$ are five times continuously differentiable sequences on Θ, $G(\boldsymbol{\theta})$ is four times continuously differentiable, and $[D^2 h(\hat{\boldsymbol{\theta}})]^{-1}$ exists. Then the following results hold:

(a) $h^*(\hat{\boldsymbol{\theta}}_N) - h(\hat{\boldsymbol{\theta}}) = -\dfrac{1}{n} G(\hat{\boldsymbol{\theta}}) + \underbrace{\dfrac{1}{n} D^1 h(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}})}_{O(n^{-3})} + O(n^{-2})$

(b) $D^1 h^*(\hat{\boldsymbol{\theta}}_N) - D^1 h(\hat{\boldsymbol{\theta}}) = O(n^{-2})$

(c)
$$
\begin{aligned}
h^*_{ij}(\hat{\boldsymbol{\theta}}_N) - h_{ij}(\hat{\boldsymbol{\theta}}) &= \frac{1}{n} D^1 h_{ij}(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}}) - \frac{1}{n} G_{ij}(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \\
&= \frac{1}{n} \sum_{\alpha,\beta} h_{\alpha ij} h^{\alpha\beta} G_\beta - \frac{1}{n} G_{ij} + O(n^{-2})
\end{aligned}
$$

(d)
$$
\begin{aligned}
h^*_{ijk}(\hat{\boldsymbol{\theta}}_N) - h_{ijk}(\hat{\boldsymbol{\theta}}) &= \frac{1}{n} D^1 h_{ijk}(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}}) - \frac{1}{n} G_{ijk}(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \\
&= \frac{1}{n} \sum_{\alpha\beta} h_{\alpha ijk} h^{\alpha\beta} G_\beta - \frac{1}{n} G_{ijk} + O(n^{-2})
\end{aligned}
$$

(e)
$$
\begin{aligned}
h^*_{ijkq}(\hat{\boldsymbol{\theta}}_N) - h_{ijkq}(\hat{\boldsymbol{\theta}}) &= \frac{1}{n} D^1 h_{ijkq}(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}}) - \frac{1}{n} G_{ijkq}(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \\
&= \frac{1}{n} \sum_{\alpha\beta} h_{\alpha ijkq} h^{\alpha\beta} G_\beta - \frac{1}{n} G_{ijkq} + O(n^{-2})
\end{aligned}
$$

(f)
$$
\begin{aligned}
h^{ij}(\hat{\boldsymbol{\theta}}_N) - h^{ij}(\hat{\boldsymbol{\theta}}) &= \frac{1}{n} D^1 h^{ij}(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \\
&= -\frac{1}{n} \sum_{\alpha\beta lm} h^{i\alpha} h_{l\alpha\beta} h^{\beta j} h^{lm} G_m + O(n^{-2})
\end{aligned}
$$

(g) $h^{*ij}(\hat{\boldsymbol{\theta}}_N) - h^{ij}(\hat{\boldsymbol{\theta}}) = -\frac{1}{n}\sum_{\alpha,\beta,l,m} h^{i\alpha}h_{l\alpha\beta}h^{\beta j}h^{lm}G_m + \frac{1}{n}\sum_{\alpha,\beta} h^{i\alpha}G_{\alpha\beta}h^{\beta j} + O(n^{-2})$

Proof of (a). From $Dh^*(\hat{\boldsymbol{\theta}}) = Dh(\hat{\boldsymbol{\theta}}) - (1/n)DG(\hat{\boldsymbol{\theta}}) = -(1/n)DG(\hat{\boldsymbol{\theta}}) + O(n^{-2})$

$$\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}} = -[D^2h^*(\hat{\boldsymbol{\theta}})]^{-1}D^1h^*(\hat{\boldsymbol{\theta}}) \tag{B.1}$$

$$= \frac{1}{n}[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \tag{B.2}$$

It follows from the result (B.2) that

$$\begin{aligned}
h^*(\hat{\boldsymbol{\theta}}_N) - h(\hat{\boldsymbol{\theta}}) &= h(\hat{\boldsymbol{\theta}}_N) - h(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G(\hat{\boldsymbol{\theta}}_N)\\
&= D^1h(\hat{\boldsymbol{\theta}})^T(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) - \frac{1}{n}G(\hat{\boldsymbol{\theta}}_N) + O(n^{-2})\\
&= D^1h(\hat{\boldsymbol{\theta}})^T\left\{\frac{1}{n}[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}})\right\} - \frac{1}{n}G(\hat{\boldsymbol{\theta}}) + O(n^{-2})\\
&= \frac{1}{n}D^1h(\hat{\boldsymbol{\theta}})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G(\hat{\boldsymbol{\theta}}) + O(n^{-2}).
\end{aligned}$$

Proof of (b). Because $\hat{\boldsymbol{\theta}}_N$ is an asymptotic mode of order n^{-2} for $-h^*$, this is immediate from Miyata (2004, p.1047).

Proof of (c). Because $D^2h^*(\hat{\boldsymbol{\theta}}_N) = D^2h(\hat{\boldsymbol{\theta}}_N) - (1/n)D^2G(\hat{\boldsymbol{\theta}}_N)$, arguing as in the proof of (a) of Lemma 13, we have

$$\begin{aligned}
h^*_{ij}(\hat{\boldsymbol{\theta}}_N) - h_{ij}(\hat{\boldsymbol{\theta}}) &= h_{ij}(\hat{\boldsymbol{\theta}}_N) - h_{ij}(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ij}(\hat{\boldsymbol{\theta}}_N)\\
&= D^1h_{ij}(\hat{\boldsymbol{\theta}})^T(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ij}(\hat{\boldsymbol{\theta}}_N) + O(n^{-2})\\
&= D^1h_{ij}(\hat{\boldsymbol{\theta}})^T\left\{\frac{1}{n}[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}})\right\} - \frac{1}{n}G_{ij}(\hat{\boldsymbol{\theta}}) + O(n^{-2})\\
&= \frac{1}{n}D^1h_{ij}(\hat{\boldsymbol{\theta}})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ij}(\hat{\boldsymbol{\theta}}) + O(n^{-2})\\
&= \frac{1}{n}\sum_{\alpha,\beta} h_{\alpha ij}h^{\alpha\beta}G_\beta(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ij}(\hat{\boldsymbol{\theta}}) + O(n^{-2}).
\end{aligned}$$

Proof of (d). Arguing as in the proof of (a) of Lemma 13,

$$\begin{aligned}
h^*_{ijk}(\hat{\boldsymbol{\theta}}_N) - h_{ijk}(\hat{\boldsymbol{\theta}}) &= h_{ijk}(\hat{\boldsymbol{\theta}}_N) - h_{ijk}(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ijk}(\hat{\boldsymbol{\theta}}_N)\\
&= D^1h_{ijk}(\hat{\boldsymbol{\theta}})^T(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ijk}(\hat{\boldsymbol{\theta}}_N) + O(n^{-2})\\
&= D^1h_{ijk}(\hat{\boldsymbol{\theta}})^T\left\{\frac{1}{n}[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}})\right\} - \frac{1}{n}G_{ijk}(\hat{\boldsymbol{\theta}}) + O(n^{-2})\\
&= \frac{1}{n}D^1h_{ijk}(\hat{\boldsymbol{\theta}})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ijk}(\hat{\boldsymbol{\theta}}) + O(n^{-2})\\
&= \frac{1}{n}\sum_{\alpha,\beta} h_{\alpha ijk}h^{\alpha\beta}G_\beta(\hat{\boldsymbol{\theta}}) - \frac{1}{n}G_{ijk}(\hat{\boldsymbol{\theta}}) + O(n^{-2}).
\end{aligned}$$

Proof of (e). This is the same as that of (d) essentially.

Proof of (f). It follows from (B.2) that

$$\begin{aligned} h^{ij}(\hat{\boldsymbol{\theta}}_N) - h^{ij}(\hat{\boldsymbol{\theta}}) &= D^1 h^{ij}(\hat{\boldsymbol{\theta}})^T (\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) + O(n^{-2}) \\ &= \frac{1}{n} D^1 h^{ij}(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \end{aligned} \tag{B.3}$$

By using the matrix algebra (D.A. Harville p.308),

$$\frac{\partial}{\partial \hat{\theta}_k} [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} = -[D^2 h(\hat{\boldsymbol{\theta}})]^{-1} \frac{\partial}{\partial \hat{\theta}_k} D^2 h(\hat{\boldsymbol{\theta}}) [D^2 h(\hat{\boldsymbol{\theta}})]^{-1}.$$

Hence,

$$\frac{\partial}{\partial \hat{\theta}_k} h^{ij}(\hat{\boldsymbol{\theta}}) = -\sum_{\alpha,\beta} h^{i\alpha} h_{k\alpha\beta} h^{\beta j}.$$

Therefore,

$$D^1 h^{ij}(\hat{\boldsymbol{\theta}}) = -\sum_{\alpha,\beta} h^{i\alpha} h^{\beta j} D^1 h_{\alpha\beta}(\hat{\boldsymbol{\theta}}).$$

Thus

$$\begin{aligned} h^{ij}(\hat{\boldsymbol{\theta}}_N) - h^{ij}(\hat{\boldsymbol{\theta}}) &= \frac{1}{n} \left(-\sum_{\alpha\beta} h^{i\alpha} h^{\beta j} D^1 h_{\alpha\beta}(\hat{\boldsymbol{\theta}})^T \right) [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 G(\hat{\boldsymbol{\theta}}) + O(n^{-2}) \\ &= -\frac{1}{n} \sum_{\alpha\beta kl} h^{i\alpha} h_{k\alpha\beta} h^{\beta j} h^{kl} G_l + O(n^{-2}). \end{aligned}$$

Proof of (g). It follows from (A.3) that

$$\begin{aligned} &[D^2 h^*(\hat{\boldsymbol{\theta}}_N)]^{-1} - [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} \\ &= [D^2 h(\hat{\boldsymbol{\theta}}_N) - \frac{1}{n} D^2 G(\hat{\boldsymbol{\theta}}_N)]^{-1} - [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} \\ &= [D^2 h(\hat{\boldsymbol{\theta}}_N)]^{-1} - [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} + \frac{1}{n} [D^2 h^*(\hat{\boldsymbol{\theta}}_N)]^{-1} D^2 G(\hat{\boldsymbol{\theta}}_N) [D^2 h(\hat{\boldsymbol{\theta}}_N)]^{-1} \\ &= [D^2 h(\hat{\boldsymbol{\theta}}_N)]^{-1} - [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} + \frac{1}{n} [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^2 G(\hat{\boldsymbol{\theta}}) [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} + O(n^{-2}). \end{aligned}$$

Using (f) of this lemma, we have

$$\begin{aligned} &h^{*ij}(\hat{\boldsymbol{\theta}}_N) - h^{ij}(\hat{\boldsymbol{\theta}}) \\ &= h^{ij}(\hat{\boldsymbol{\theta}}_N) - h^{ij}(\hat{\boldsymbol{\theta}}) + \frac{1}{n} \begin{pmatrix} h^{i1} & \cdots & h^{id} \end{pmatrix} D^2 G(\hat{\boldsymbol{\theta}}) \begin{pmatrix} h^{1j} \\ \vdots \\ h^{dj} \end{pmatrix} + O(n^{-2}) \\ &= -\frac{1}{n} \sum_{\alpha\beta lm} h^{i\alpha} h_{l\alpha\beta} h^{\beta j} h^{lm} G_m + \frac{1}{n} \sum_{\alpha\beta} h^{i\alpha} G_{\alpha\beta} h^{\beta j} + O(n^{-2}). \end{aligned}$$

The following expansions are used to derive the fully exponential Laplace approximations in Section 3.

Lemma 15. Suppose that $\hat{\boldsymbol{\theta}}$ is an asymptotic mode of order n^{-2} for $-h(\hat{\boldsymbol{\theta}})$. Let μ^*_{ijkqrs} be sixth central moments of a multivariate Normal distribution with mean $\mathbf{0}$ and covariance $\Sigma^* = [nD^2h^*(\hat{\boldsymbol{\theta}}_N)]^{-1}$. Let

$$\Delta^{k_5k_6} = \sum_{\alpha\beta lm} h^{k_5\alpha}h_{l\alpha\beta}h^{\beta k_6}h^{lm}G_m - \sum_{\alpha\beta} h^{k_5\alpha}h^{\beta k_6}G_{\alpha\beta}.$$

Then the following results hold.

(a) $n^3(\mu^*_{ijkqrs} - \mu_{ijkqrs}) = -\dfrac{1}{n}\displaystyle\sum_{<k_1,k_2\cdots k_6>} h^{k_1k_2}h^{k_3k_4}\Delta^{k_5k_6} + O(n^{-2}),$

where $\mathcal{A}$, $\mathcal{A}$, and $\mathcal{B}$ are three rooms in which two rooms are the same, and the other is different from them, and $< k_1, k_2, ..., k_6 >$ denotes ways to arrange elements i, j, k, q, r and s into the three rooms $\mathcal{A}$, $\mathcal{A}$, and $\mathcal{B}$ by two alphabets. Hence the summation $\sum_{<k_1,k_2\cdots k_6>}$ is over $\binom{6}{2}\cdot\binom{4}{2}/2 = 45$ terms.

(b)

$$\begin{aligned}
&\sum_{ijkqrs} h_{ijk}h_{qrs}n^3(\mu^*_{ijkqrs} - \mu_{ijkqrs}) \\
&= -\frac{9}{n}\sum_{ijkqrs\alpha\beta lm} h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{9}{n}\sum_{ijkqrs\alpha\beta} h_{ijk}h_{qrs}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}G_{\alpha\beta} \\
&- \frac{18}{n}\sum_{ijkqrs\alpha\beta lm} h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{18}{n}\sum_{ijkqrs\alpha\beta} h_{ijk}h_{qrs}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}G_{\alpha\beta} \\
&- \frac{18}{n}\sum_{ijkqrs\alpha\beta lm} h_{ijk}h_{qrs}h_{l\alpha\beta}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}h^{lm}G_m + \frac{18}{n}\sum_{ijkqrs\alpha\beta} h_{ijk}h_{qrs}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}G_{\alpha\beta} \\
&+ O(n^{-2}).
\end{aligned}$$

(c) $n^3\displaystyle\sum_{ijkqrs}\{h^*_{ijk}(\hat{\boldsymbol{\theta}}_N) - h_{ijk}(\hat{\boldsymbol{\theta}})\}h_{qrs}\mu^*_{ijkqrs}$

$$= \frac{1}{n}\sum_{ijkqrs\alpha\beta} h_{\alpha ijk}h_{qrs}h^{\alpha\beta}(n^3\mu_{ijkqrs})G_\beta - \sum_{ijkqrs} h_{qrs}(n^3\mu_{ijkqrs})G_{ijk} + O(n^{-2}).$$

(d) $h^*_l(\hat{\boldsymbol{\theta}}_N) = \dfrac{1}{2n^2}\displaystyle\sum_{rm\alpha\beta} h_{l\alpha\beta}h^{\alpha r}h^{\beta m}G_rG_m + O(n^{-3}).$

Proof of (a). It is well known that sixth central moments of the Normal distribution are decomposed with elements of the covariance matrix i.e.,

$$\mu^*_{ijkqrs} = \frac{1}{n^3}\Bigg(\underbrace{h^{*ij}h^{*kq}h^{*rs} + h^{*ij}h^{*kr}h^{*qs} + \cdots + h^{*is}h^{*jr}h^{*kq}}_{15 \text{ terms}}\Bigg), \tag{B.4}$$

where h^{*ij} are (i, j)-elements of $[D^2h^*(\hat{\boldsymbol{\theta}}_N)]^{-1}$. For example, see Kass et al. (1990, p.477). From (g) of Lemma 13, $h^{*ij}(\hat{\boldsymbol{\theta}}_N) = h^{ij}(\hat{\boldsymbol{\theta}}) - \dfrac{\Delta^{ij}}{n} + O(n^{-2})$.

Then we have

$$
\begin{aligned}
&h^{*ij}(\hat{\boldsymbol{\theta}}_N)h^{*kq}(\hat{\boldsymbol{\theta}}_N)h^{*rs}(\hat{\boldsymbol{\theta}}_N)\\
&= h^{ij}(\hat{\boldsymbol{\theta}})h^{kq}(\hat{\boldsymbol{\theta}})h^{rs}(\hat{\boldsymbol{\theta}})\\
&-\frac{1}{n}h^{ij}(\hat{\boldsymbol{\theta}})h^{kq}(\hat{\boldsymbol{\theta}})\Big(\sum_{\alpha\beta lm} h^{r\alpha}h_{l\alpha\beta}h^{\beta s}h^{lm}G_m - \sum_{\alpha\beta} h^{r\alpha}h^{\beta s}G_{\alpha\beta}\Big)\\
&-\frac{1}{n}h^{ij}(\hat{\boldsymbol{\theta}})h^{rs}(\hat{\boldsymbol{\theta}})\Big(\sum_{\alpha\beta lm} h^{k\alpha}h_{l\alpha\beta}h^{\beta q}h^{lm}G_m - \sum_{\alpha\beta} h^{k\alpha}h^{\beta q}G_{\alpha\beta}\Big)\\
&-\frac{1}{n}h^{kq}(\hat{\boldsymbol{\theta}})h^{rs}(\hat{\boldsymbol{\theta}})\Big(\sum_{\alpha\beta lm} h^{i\alpha}h_{l\alpha\beta}h^{\beta j}h^{lm}G_m - \sum_{\alpha\beta} h^{i\alpha}h^{\beta j}G_{\alpha\beta}\Big) + O(n^{-2})\\
&= h^{ij}(\hat{\boldsymbol{\theta}})h^{kq}(\hat{\boldsymbol{\theta}})h^{rs}(\hat{\boldsymbol{\theta}}) - \frac{1}{n}h^{ij}(\hat{\boldsymbol{\theta}})h^{kq}(\hat{\boldsymbol{\theta}})\Delta^{rs} - \frac{1}{n}h^{ij}(\hat{\boldsymbol{\theta}})h^{rs}(\hat{\boldsymbol{\theta}})\Delta^{kq}\\
&\qquad - \frac{1}{n}h^{kq}(\hat{\boldsymbol{\theta}})h^{rs}(\hat{\boldsymbol{\theta}})\Delta^{ij} + O(n^{-2}),
\end{aligned}
\tag{B.5}
$$

where

$$
\Delta^{ij} = \sum_{\alpha,\beta,l,m} h^{i\alpha}h_{l\alpha\beta}h^{\beta j}h^{lm}G_m - \sum_{\alpha,\beta} h^{i\alpha}h^{\beta j}G_{\alpha\beta}. \tag{B.6}
$$

Substitution of (B.5) into the decomposition (B.4) leads to the equation (a).
Proof of (b). From Lemma 14 (a), we have

$$
\begin{aligned}
&\sum_{ijkqrs} h_{ijk}h_{qrs}n^3(\mu^*_{ijkqrs} - \mu_{ijkqrs})\\
&= -\frac{1}{n}\sum_{ijkqrs} h_{ijk}h_{qrs}\Big(\underbrace{h^{ij}h^{kq}\Delta^{rs} + h^{ij}\Delta^{kq}h^{rs} + \Delta^{ij}h^{kq}h^{rs} + \cdots + h^{is}h^{jr}\Delta^{kq}}_{45 terms}\Big) + O(n^{-2})\\
&= -\frac{1}{n}\sum_{ijkqrs} h_{ijk}h_{qrs}\sum_{<k_1k_2\cdots k_6>} h^{k_1k_2}h^{k_3k_4}\Delta^{k_5k_6} + O(n^{-2}).
\end{aligned}
\tag{B.7}
$$

Equation (B.7) are divided into three kinds of terms, $\sum h_{ijk}h_{qrs}\Delta^{iq}h^{jk}h^{rs}$, $\sum h_{ijk}h_{qrs}\Delta^{iq}h^{jr}h^{ks}$, and $\sum h_{ijk}h_{qrs}\Delta^{ij}h^{kq}h^{rs}$ as described below. Note that $\sum h_{ijk}h_{qrs}h^{ij}h^{kq}\Delta^{rs}$ and $\sum h_{ijk}h_{qrs}\Delta^{ij}h^{kq}h^{rs}$ are the same if the suffixes i and j are exchanged for r and s. The second summation in (B.7) is

$$
\begin{aligned}
&\sum_{<k_1 k_2 \cdots k_6>} h^{k_1k_2} h^{k_3k_4} \triangle^{k_5k_6} \\
&= h^{ij}h^{kq}\triangle^{rs} + h^{ij}h^{kr}\triangle^{qs} + h^{ij}h^{ks}\triangle^{qr} + h^{ik}h^{jq}\triangle^{rs} + h^{ik}h^{jr}\triangle^{qs} + h^{ik}h^{js}\triangle^{qr} \\
&+ h^{iq}h^{jk}\triangle^{rs} + \underline{h^{iq}h^{jr}\triangle^{ks} + h^{iq}h^{js}\triangle^{kr}} + h^{ir}h^{jk}\triangle^{qs} + \underline{h^{ir}h^{jq}\triangle^{ks} + h^{ir}h^{js}\triangle^{kq}} \\
&+ h^{is}h^{jk}\triangle^{qr} + \underline{h^{is}h^{jq}\triangle^{kr} + h^{is}h^{jr}\triangle^{kq}} + \underbrace{h^{ij}\triangle^{kq}h^{rs} + h^{ij}\triangle^{kr}h^{qs} + h^{ij}\triangle^{ks}h^{qr}} \\
&+ \underbrace{h^{ik}\triangle^{jq}h^{rs} + h^{ik}\triangle^{jr}h^{qs} + h^{ik}\triangle^{js}h^{qr}} + h^{iq}\triangle^{jk}h^{rs} + \underline{h^{iq}\triangle^{jr}h^{ks} + h^{iq}\triangle^{js}h^{kr}} \\
&+ h^{ir}\triangle^{jk}h^{qs} + \underline{h^{ir}\triangle^{jq}h^{ks} + h^{ir}\triangle^{js}h^{kq}} + h^{is}\triangle^{jk}h^{qr} + \underline{h^{is}\triangle^{jq}h^{kr} + h^{is}\triangle^{jr}h^{kq}} \\
&+ \triangle^{ij}h^{kq}h^{rs} + \triangle^{ij}h^{kr}h^{qs} + \triangle^{ij}h^{ks}h^{qr} + \triangle^{ik}h^{jq}h^{rs} + \triangle^{ik}h^{jr}h^{qs} + \triangle^{ik}h^{js}h^{qr} \\
&+ \underbrace{\triangle^{iq}h^{jk}h^{rs}} + \underline{\triangle^{iq}h^{jr}h^{ks} + \triangle^{iq}h^{js}h^{kr}} + \underbrace{\triangle^{ir}h^{jk}h^{qs}} + \underline{\triangle^{ir}h^{jq}h^{ks} + \triangle^{ir}h^{js}h^{kq}} \\
&+ \underbrace{\triangle^{is}h^{jk}h^{qr}} + \underline{\triangle^{is}h^{jq}h^{kr} + \triangle^{is}h^{jr}h^{kq}}.
\end{aligned}
$$

Then we classify the foregoing equation into three type listed below.
① (The underline) In the suffixes k_5 and k_6 of $\Delta^{k_5k_6}$, one is from i, j and k, and another is from q, r and s. Additionally, in k_3 and k_4 of $h^{k_3k_4}$, one is from i, j and k, and another is from q, r and s. (全部で 18 通り)

② (The underbrace) In the suffixes k_5 and k_6 of $\Delta^{k_5k_6}$, one is from i, j and k, and another is from q, r and s. Additionally, the suffixes k_3 and k_4 have two alphabets in either (i, j, k) or (q, r, s). (全部で 9 通り)

③ The suffixes k_5 and k_6 of $\Delta^{k_5k_6}$ have two alphabets in either (i, j, k) or (q, r, s). (全部で 18 通り)

Consequently,

$$
\begin{aligned}
&\sum_{ijkqrs} h_{ijk}h_{qrs}n^3(\mu^*_{ijkqrs} - \mu_{ijkqrs}) \\
&= -\frac{9}{n}\sum h_{ijk}h_{qrs}\Delta^{iq}h^{jk}h^{rs} - \frac{18}{n}\sum h_{ijk}h_{qrs}\Delta^{iq}h^{jr}h^{ks} \\
&\quad - \frac{18}{n}\sum h_{ijk}h_{qrs}\Delta^{ij}h^{kq}h^{rs} + O(n^{-2}) \\
&= -\frac{9}{n}\sum h_{ijk}h_{qrs}h^{jk}h^{rs}\left(\sum_{\alpha,\beta,l,m} h^{i\alpha}h_{l\alpha\beta}h^{\beta q}h^{lm}G_m - \sum_{\alpha,\beta} h^{i\alpha}G_{\alpha\beta}h^{\beta q}\right) \\
&- \frac{18}{n}\sum_{ijkqrs} h_{ijk}h_{qrs}h^{jr}h^{ks}\left(\sum_{\alpha,\beta,l,m} h^{i\alpha}h_{l\alpha\beta}h^{\beta q}h^{lm}G_m - \sum_{\alpha,\beta} h^{i\alpha}G_{\alpha\beta}h^{\beta q}\right) \\
&- \frac{18}{n}\sum_{ijkqrs} h_{ijk}h_{qrs}h^{kq}h^{rs}\left(\sum_{\alpha,\beta,l,m} h^{i\alpha}h_{l\alpha\beta}h^{\beta j}h^{lm}G_m - \sum_{\alpha,\beta} h^{i\alpha}G_{\alpha\beta}h^{\beta j}\right) + O(n^{-2}).
\end{aligned}
$$

Thus the foregoing equation becomes

$$
\begin{aligned}
&-\frac{9}{n}\sum_{ijkqrs\alpha\beta lm} h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{9}{n}\sum_{ijkqrs\alpha\beta} h_{ijk}h_{qrs}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}G_{\alpha\beta} \\
&-\frac{18}{n}\sum_{ijkqrs\alpha\beta lm} h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{18}{n}\sum_{ijkqrs\alpha\beta} h_{ijk}h_{qrs}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}G_{\alpha\beta} \\
&-\frac{18}{n}\sum_{ijkqrs\alpha\beta lm} h_{ijk}h_{qrs}h_{l\alpha\beta}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}h^{lm}G_m + \frac{18}{n}\sum_{ijkqrs\alpha\beta} h_{ijk}h_{qrs}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}G_{\alpha\beta} \\
&+O(n^{-2}).
\end{aligned}
$$

Proof of (c). It follows from Lemma 13 (d) that

$$
\begin{aligned}
&n^3\sum_{ijkqrs}(h^*_{ijk}(\hat{\boldsymbol{\theta}}_N) - h_{ijk}(\hat{\boldsymbol{\theta}}))h_{qrs}\mu^*_{ijkqrs} \\
&= n^3\sum_{ijkqrs}\left(\frac{1}{n}\sum_{\alpha,\beta} h_{\alpha ijk}h^{\alpha\beta}G_\beta - \frac{1}{n}G_{ijk} + O(n^{-2})\right)h_{qrs}\left(\mu_{ijkqrs} + O(n^{-4})\right) \\
&= \frac{1}{n}\sum_{ijkqrs\alpha\beta} h_{\alpha ijk}h_{qrs}h^{\alpha\beta}(n^3\mu_{ijkqrs})G_\beta - \sum_{ijkqrs} h_{qrs}(n^3\mu_{ijkqrs})G_{ijk} + O(n^{-2})
\end{aligned}
$$

Hence, (c) is proved.

Proof of (d) Expanding the first derivatives $h^*_l(\hat{\boldsymbol{\theta}}_N)$ around $\hat{\boldsymbol{\theta}}$, and using (B.1) and (B.2) yields

$$
\begin{aligned}
h^*_l(\hat{\boldsymbol{\theta}}_N) &= h^*_l(\hat{\boldsymbol{\theta}}) + D^1h^*_l(\hat{\boldsymbol{\theta}})^T(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) + \frac{1}{2}(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}})^T D^2h^*_l(\hat{\boldsymbol{\theta}})(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) + O(n^{-3}) \\
&= h^*_l(\hat{\boldsymbol{\theta}}) + D^1h^*_l(\hat{\boldsymbol{\theta}})^T\left(-[D^2h^*(\hat{\boldsymbol{\theta}})]^{-1}D^1h^*(\hat{\boldsymbol{\theta}})\right) \\
&+ \frac{1}{2}\left(\frac{1}{n}[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}})\right)^T D^2h^*_l(\hat{\boldsymbol{\theta}})\left(\frac{1}{n}[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}})\right) + O(n^{-3}).
\end{aligned}
$$

It follows from definition of an inverse matrix that

$$
D^1h^*_l(\hat{\boldsymbol{\theta}})^T[D^2h^*(\hat{\boldsymbol{\theta}})]^{-1} = (0, ..., 0, \underset{l}{1}, 0, ..., 0).
$$

Thus we have

$$
\begin{aligned}
h^*_l(\hat{\boldsymbol{\theta}}_N) &= h^*_l(\hat{\boldsymbol{\theta}}) - h^*_l(\hat{\boldsymbol{\theta}}) + \frac{1}{2n^2}D^1G(\hat{\boldsymbol{\theta}})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^2h_l(\hat{\boldsymbol{\theta}})[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}}) + O(n^{-3}) \\
&= \frac{1}{2n^2}D^1G(\hat{\boldsymbol{\theta}})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^2h_l(\hat{\boldsymbol{\theta}})[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1G(\hat{\boldsymbol{\theta}}) + O(n^{-3}) \\
&= \frac{1}{2n^2}\left(\sum_r h^{1r}G_r \cdots \sum_r h^{dr}G_r\right)D^2h_l(\hat{\boldsymbol{\theta}})\begin{pmatrix}\sum_q h^{1q}G_q \\ \vdots \\ \sum_q h^{dq}G_q\end{pmatrix} + O(n^{-3}) \\
&= \frac{1}{2n^2}\sum_{l\alpha\beta qr} h_{l\alpha\beta}h^{\alpha r}h^{\beta q}G_rG_q + O(n^{-3}).
\end{aligned}
$$

Hence, the lemma is proved.

Appendix C: Proofs of the main results

This section proves the main results. The notations are the same as in Sections 2 and 3.

Proof of Theorem 1.

$$\int_{\boldsymbol{\Theta}} e^{-nh(\boldsymbol{\theta})} d\boldsymbol{\theta} = \int_{B_\delta(\boldsymbol{\theta})} e^{-nh(\boldsymbol{\theta})} d\boldsymbol{\theta} + \int_{\boldsymbol{\Theta}-B_\delta(\boldsymbol{\theta})} e^{-nh(\boldsymbol{\theta})} d\boldsymbol{\theta}.$$

First, we evaluate the first term in the r.h.s.

By expanding $h(\boldsymbol{\theta})$ about $\hat{\boldsymbol{\theta}}$,

$$\begin{aligned} h(\boldsymbol{\theta}) = h(\hat{\boldsymbol{\theta}}) &+ \sum_i h_i(\hat{\boldsymbol{\theta}}) z_i + \frac{1}{2}\sum_{ij} h_{ij}(\hat{\boldsymbol{\theta}}) z_i z_j + \frac{1}{6}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\theta}}) z_i z_j z_k \\ &+ \frac{1}{24}\sum_{ijkq} h_{ijkq}(\hat{\boldsymbol{\theta}}) z_i z_j z_k z_q + \frac{1}{120}\sum h_{ijkqr}(\hat{\boldsymbol{\theta}}) z_i z_j z_k z_q z_r \\ &+ \frac{1}{720}\sum h_{ijkqrs}(\hat{\boldsymbol{\theta}}) z_i z_j z_k z_q z_r z_s + R_{1n}, \end{aligned}$$

where $R_{1n} \equiv (1/7!) \sum h_{ijkqrst} z_i z_j z_k z_q z_r z_s z_t + r_{1n}$, and r_{1n} is bounded over $B_\epsilon(\hat{\boldsymbol{\theta}})$ by a polynomial in $z_i z_j z_k z_q z_r z_s z_t z_u$. By using Lemma 7, we have

$$\begin{aligned} &\sum_i h_i(\hat{\boldsymbol{\theta}}) z_i + \frac{1}{2}\sum_{ij} h_{ij}(\hat{\boldsymbol{\theta}}) z_i z_j \\ &= D^1 h(\hat{\boldsymbol{\theta}})^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \frac{1}{2}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T D^2 h(\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) \\ &= \frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T D^2 h(\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\mathbf{y}) - \frac{1}{4} D^1 h(\hat{\boldsymbol{\theta}})^T [\frac{1}{2} D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 h(\hat{\boldsymbol{\theta}}), \end{aligned}$$

where $\mathbf{y} = \hat{\boldsymbol{\theta}} - [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 h(\hat{\boldsymbol{\theta}})$.

Using the expansion $e^x = 1 + x + (1/2)x^2 + (1/6)x^3 + (1/24)x^4 + (1/120)e^{\tau_1} x^5$, where τ_1 is a point between 0 and x, it follows that

$$\begin{aligned} &exp\left\{-n\left[\frac{1}{6}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\theta}}) z_i z_j z_k + \cdots + \frac{1}{720}\sum h_{ijkqrs}(\hat{\boldsymbol{\theta}}) z_i z_j z_k z_q z_r z_s + R_{1n}\right]\right\} \\ &= \Bigg\{1 - n\bigg(\frac{1}{6}\sum h_{ijk} z_i z_j z_k + \frac{1}{24}\sum h_{ijkq} z_i z_j z_k z_q + \frac{1}{120}\sum h_{ijkqr} z_i z_j z_k z_q z_r \\ &\qquad\qquad + \frac{1}{720}\sum h_{ijkqrs} z_i z_j z_k z_q z_r z_s + R_{1n}\bigg) \\ &+ \frac{n^2}{2}\bigg(\frac{1}{36}[\sum_{ijk} h_{ijk} z_i z_j z_k]^2 + 2\cdot\frac{1}{6}\cdot\frac{1}{24}[\sum_{ijk} h_{ijk} z_i z_j z_k][\sum_{qrst} h_{qrst} z_q z_r z_s z_t] \\ &\quad + \left(\frac{1}{24}\right)^2 [\sum h_{ijkq} z_i z_j z_k z_q]^2 + 2\cdot\frac{1}{6}\cdot\frac{1}{120}[\sum h_{ijk} z_i z_j z_k][\sum_{qrstu} h_{qrstu} z_q z_r z_s z_t z_u] + R_{2n}\bigg) \\ &- \frac{n^3}{6}\bigg(\left(\frac{1}{6}\right)^3 [\sum_{ijk} h_{ijk} z_i z_j z_k]^3 + \binom{3}{2}\cdot\left(\frac{1}{6}\right)^2 \cdot \frac{1}{24}[\sum_{ijk} h_{ijk} z_i z_j z_k]^2 [\sum_{qrst} h_{qrst} z_q z_r z_s z_t] + R_{3n}\bigg) \\ &\quad + \frac{n^4}{24}\bigg(\left(\frac{1}{6}\right)^4 [\sum_{ijk} h_{ijk} z_i z_j z_k]^4 + R_{4n}\bigg) + R_{5n}\Bigg\} \\ &= J_n(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}) + R_n, \end{aligned} \tag{C.1}$$

where R_{2n}, R_{3n}, and R_{4n} are the rests of terms squared, cubed, and to the fourth power in the bracket, R_{5n} is a term with polynomials of degree greater than or equal to 15, and R_n is the sum of all terms involving R_{1n}, R_{2n}, R_{3n}, R_{4n}, and R_{5n}.

In (C.1), $J_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})$ is

$$\begin{aligned}
J_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) = 1 &- \frac{n}{6}\sum h_{ijk}z_iz_jz_k - \frac{n}{24}\sum h_{ijkq}z_iz_jz_kz_q \\
&+ \frac{n^2}{72}\sum h_{ijk}h_{qrs}z_iz_jz_kz_qz_rz_s \\
&- \frac{n}{120}\sum h_{ijkqr}z_iz_jz_kz_qz_r - \frac{n}{720}\sum h_{ijkqrs}z_iz_jz_kz_qz_rz_s \\
&+ \frac{n^2}{144}\sum h_{ijk}h_{qrst}z_iz_jz_kz_qz_rz_sz_t + \frac{n^2}{1152}\sum h_{ijkq}h_{rstu}z_iz_jz_kz_qz_rz_sz_tz_u \\
&+ \frac{n^2}{720}\sum h_{ijk}h_{qrstu}z_iz_jz_kz_qz_rz_sz_tz_u \\
&- \frac{n^3}{1296}\sum h_{ijk}h_{qrs}h_{tuv}z_iz_jz_kz_qz_rz_sz_tz_uz_v \\
&- \frac{n^3}{1728}\sum h_{ijk}h_{qrs}h_{tuvw}z_iz_jz_kz_qz_rz_sz_tz_uz_vz_w \\
&+ \frac{n^4}{31104}\sum h_{ijk}h_{qrs}h_{tuv}h_{wxy}z_iz_jz_kz_qz_rz_sz_tz_uz_vz_wz_xz_y.
\end{aligned}$$

From (C.1), we have

$$\begin{aligned}
&\int_{B_\delta(\hat{\boldsymbol{\theta}})} e^{-nh(\boldsymbol{\theta})}d\boldsymbol{\theta} \\
&= e^{-nh(\hat{\boldsymbol{\theta}})}\cdot exp\left\{\frac{n}{4}D^1h(\hat{\boldsymbol{\theta}})^T[\frac{1}{2}D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1h(\hat{\boldsymbol{\theta}})\right\} \\
&\times \int_{B_\delta(\hat{\boldsymbol{\theta}})} exp\left(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T(\frac{1}{n}D^2h(\hat{\boldsymbol{\theta}})^{-1})^{-1}(\boldsymbol{\theta}-\mathbf{y})\right) \\
&\times exp\left\{-n\left(\frac{1}{6}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\theta}})z_iz_jz_k + \cdots + \frac{1}{720}\sum h_{ijkqrs}z_iz_jz_kz_qz_rz_s + R_{1n}\right)\right\}d\boldsymbol{\theta}. \\
&= e^{-nh(\hat{\boldsymbol{\theta}})}C_n(\hat{\boldsymbol{\theta}})\int_{B_\delta(\hat{\boldsymbol{\theta}})} exp\left(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\right)J_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})d\boldsymbol{\theta}, \qquad \text{(C.2)} \\
&+ e^{-nh(\hat{\boldsymbol{\theta}})}C_n(\hat{\boldsymbol{\theta}})\int_{B_\delta(\hat{\boldsymbol{\theta}})} exp\left(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\right)R_nd\boldsymbol{\theta} \qquad \text{(C.3)}
\end{aligned}$$

where $\Sigma = [nD^2h(\hat{\boldsymbol{\theta}})]^{-1}$. Second, we evaluate (C.3). The terms composing R_n may be represented explicitly using the mean value form of the remainders in terms of higher derivatives of h evaluated at points between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$, for example, one such term is
$(-n/8!)\sum h_{ijkqrstu}(\gamma_1)z_iz_jz_kz_qz_rz_sz_tz_u$, where γ_1 is a point between $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$. It is one piece of the error term appearing as R_n. Because it follows from condition (A3) that $||h_{ijkqrstu}(\gamma_1)|| < M$

on $B_\delta(\hat{\boldsymbol{\theta}})$, we have

$$\begin{aligned}
&\left|\left|\frac{-n}{8!}\int_{B_\delta(\hat{\boldsymbol{\theta}})} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)h_{ijkqrstu}(\gamma_1)z_iz_jz_kz_qz_rz_sz_tz_u d\boldsymbol{\theta}\right|\right| \\
&\leq \frac{nM}{8!}\int_{B_\delta(\hat{\boldsymbol{\theta}})} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)||z_iz_jz_kz_qz_rz_sz_tz_u||d\boldsymbol{\theta} \\
&= |\Sigma|^{1/2}\times O(n^{-3}).
\end{aligned} \tag{C.4}$$

The other terms are similar. Thus (C.3) becomes $(2\pi)^{d/2}e^{-nh(\hat{\boldsymbol{\theta}})}|\Sigma|^{1/2}C_n(\hat{\boldsymbol{\theta}})\times O(n^{-3})$.
Third, we evaluate (C.2). Since there exists a symmetric matrix $\mathbf{A}^{1/2}$ such that $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = D^2h(\hat{\boldsymbol{\theta}})$, putting $n^{1/2}\mathbf{A}^{1/2}(\boldsymbol{\theta}-\mathbf{y}) = \mathbf{u}$, we have

$$\begin{aligned}
\boldsymbol{\Theta} - B_\delta(\hat{\boldsymbol{\theta}}) &= \{\boldsymbol{\theta} : (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) > \delta^2\} \\
&= \{\boldsymbol{\theta} : (\mathbf{u}-n^{1/2}\mathbf{A}^{1/2}\mathbf{b})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}(\mathbf{u}-n^{1/2}\mathbf{A}^{1/2}\mathbf{b}) > n\delta^2\} \\
&\subseteq \{\boldsymbol{\theta} : \frac{1}{\lambda_1}(\mathbf{u}-n^{1/2}\mathbf{A}^{1/2}\mathbf{b})^T(\mathbf{u}-n^{1/2}\mathbf{A}^{1/2}\mathbf{b}) > n\delta^2\} \\
&\subseteq \{\boldsymbol{\theta} : 2\mathbf{u}^T\mathbf{u} + 2n\mathbf{b}^T\mathbf{A}^{1/2}\mathbf{A}^{1/2}\mathbf{b} > n\lambda_1\delta^2\} \\
&= \{\boldsymbol{\theta} : \mathbf{u}^T\mathbf{u} > nc_2\} \\
&= \{\boldsymbol{\theta} : (\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y}) > nc_2\},
\end{aligned}$$

where λ_1 is the smallest eigenvalue of $D^2h(\hat{\boldsymbol{\theta}})$, $\mathbf{b} = (b_i) \equiv -[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1h(\hat{\boldsymbol{\theta}})$ and $c_2 = (1/2)\{\lambda_1\delta^2 - 2D^1h(\hat{\boldsymbol{\theta}})^T[D^2h(\hat{\boldsymbol{\theta}})]^{-1}D^1h(\hat{\boldsymbol{\theta}})\}$. Note that $c_2 > 0$ for large n because λ_1 is strictly positive from the assumption (A4). Hence, once again putting $n^{1/2}\mathbf{A}^{1/2}(\boldsymbol{\theta}-y) = \mathbf{u} = (u_i)$, we have

$$\begin{aligned}
&\int_{\boldsymbol{\Theta}-B_\delta(\hat{\boldsymbol{\theta}})} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)J_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})d\boldsymbol{\theta} \\
&\leq \left(\int_{(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})>nc_2} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)J_n(\boldsymbol{\theta}^2,\hat{\boldsymbol{\theta}})d\boldsymbol{\theta}\right)^{1/2} \\
&= |\Sigma|^{1/2}\left(\int_{\mathbf{u}^T\mathbf{u}>nc_2} exp\Big(-\frac{1}{2}\mathbf{u}^T\mathbf{u}\Big)Po(\mathbf{u})d\mathbf{u}\right)^{1/2},
\end{aligned}$$

where $Po(\mathbf{u})$ denotes a term with multivariate polynomials in $u_1, ..., u_d$ of finite degree. Consequently,

$$\int_{\boldsymbol{\Theta}-B_\delta(\hat{\boldsymbol{\theta}})} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)J_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})d\boldsymbol{\theta}.$$

becomes the term of exponentially decreasing error by the same argument as in Kass et al. (1990, pp.478-479). This allows the replacement of the domain in (C.2) by the whole Euclidean space. Therefore, (C.2) is replaced with

$$e^{-nh(\hat{\boldsymbol{\theta}})}C_n(\hat{\boldsymbol{\theta}})\int_{\mathbb{R}^d} exp\Big(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{y})^T\Sigma^{-1}(\boldsymbol{\theta}-\mathbf{y})\Big)J_n(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})d\boldsymbol{\theta}.$$

Since $\Sigma = (h^{ij}/n)$, it follows from Lemma 12 (a) that

$$nE^N[(\Theta_i-\hat{\theta}_i)(\Theta_j-\hat{\theta}_j)(\Theta_k-\hat{\theta}_k)] = b_ih^{jk} + b_jh^{ik} + b_kh^{ij} + nb_ib_jb_k.$$

Then, using $D^1 h(\hat{\boldsymbol{\theta}}) = O(n^{-1})$, we have

$$E^N\left[-\frac{n}{6}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\theta}}) z_i z_j z_k\right] = -\frac{1}{2n}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\theta}})(nb_i)h^{jk} - \frac{n}{6}\sum_{ijk} h_{ijk} b_i b_j b_k. \tag{C.5}$$

Moreover,

$$\begin{aligned}
E^N\left[-\frac{n}{24}\sum_{ijkq} h_{ijkq} z_i z_j z_k z_q\right] &= -\frac{n}{24}\sum h_{ijkq} E[z_i z_j z_k z_q] \\
= -\frac{n}{24}\sum_{ijkq} h_{ijkq}\Bigg(\frac{h^{ij}}{n}\cdot\frac{h^{kq}}{n} + \frac{h^{ik}}{n}\cdot\frac{h^{jq}}{n} + \frac{h^{iq}}{n}\cdot\frac{h^{kj}}{n} &+ \sum_{[m_1,m_2]_4} b_{m_1} b_{m_2}\frac{h^{m_3 m_4}}{n} + b_i b_j b_k b_q\Bigg) \\
= -\frac{1}{8n}\sum_{ijkq} h_{ijkq} h^{ij} h^{kq} - \frac{n}{24}\cdot\frac{1}{n}\cdot\binom{4}{2}\sum_{ijkq} h_{ijkq} h^{ij} b_k b_q &- \frac{n}{24}\sum_{ijkq} h_{ijkq} b_i b_j b_k b_q \\
= -\frac{1}{8n}\sum_{ijkq} h_{ijkq} h^{ij} h^{kq} - \frac{1}{4}\sum_{ijkq} h_{ijkq} h^{ij} b_k b_q &+ O(n^{-3})
\end{aligned} \tag{C.6}$$

Similarly, applying (iii)–(xi) of Appendix E to the other expanded terms in $J_n(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}})$, and then combining (C.5) and (C.6) yields

$$\int_{B_\delta(\hat{\boldsymbol{\theta}})} e^{-nh(\boldsymbol{\theta})} d\boldsymbol{\theta} = (2\pi)^{d/2}|\Sigma|^{1/2} e^{-nh(\hat{\boldsymbol{\theta}})} C_n(\hat{\boldsymbol{\theta}})\left(1 + \frac{\lambda_{1n}}{n} + \frac{\lambda_{2n}}{n^2} + O(n^{-3})\right), \tag{C.7}$$

where $C_n(\hat{\boldsymbol{\theta}}) = exp\{(n/2) D^1 h(\hat{\boldsymbol{\theta}})^T [D^2 h(\hat{\boldsymbol{\theta}})]^{-1} D^1 h(\hat{\boldsymbol{\theta}})\}$,

$$\lambda_{1n} = -\frac{1}{2}\sum_{ijk} h_{ijk}(\hat{\boldsymbol{\theta}})(nb_i)h^{jk} - \frac{1}{8}\sum_{ijkq} h_{ijkq}(\hat{\boldsymbol{\theta}}) h^{ij} h^{kq} + \frac{1}{72}\sum_{ijkqrs} h_{ijk}(\hat{\boldsymbol{\theta}}) h_{qrs}(\hat{\boldsymbol{\theta}})\mu_{ijkqrs} n^3,$$

and μ_{ijkqrs} are the sixth central moments of a multivariate Normal distribution having covariance matrix $[nD^2 h(\hat{\boldsymbol{\theta}})]^{-1}$.
On the other hand, it follows from assumption (A.5) of Section 2 that

$$\int_{\boldsymbol{\Theta} - B_\delta(\hat{\boldsymbol{\theta}})} e^{-nh(\boldsymbol{\theta})} d\boldsymbol{\theta} = (2\pi)^{d/2}|\Sigma|^{1/2} e^{-nh(\hat{\boldsymbol{\theta}})} C_n(\hat{\boldsymbol{\theta}}) \times O(n^{-3}). \tag{C.8}$$

Therefore, combining (C.7) and (C.8) yields the result.

Lemma 16. Suppose that $\hat{\boldsymbol{\theta}}$ is an asymptotic mode of order n^{-2} for $-h$. Then it follows that $D^1 h^*(\hat{\boldsymbol{\theta}}_N) = O(n^{-2})$.

Proof. Expanding $\partial h^*(\hat{\boldsymbol{\theta}}_N)/\partial\theta_i$ around $\hat{\boldsymbol{\theta}}$ yields, for $i = 1, ..., d$,

$$\frac{\partial}{\partial\theta_i} h^*(\hat{\boldsymbol{\theta}}_N) = \frac{\partial}{\partial\theta_i} h^*(\hat{\boldsymbol{\theta}}) + (D^1\frac{\partial}{\partial\theta_i} h^*(\hat{\boldsymbol{\theta}}))^T(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) + (\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}})^T\left[D^2\frac{\partial}{\partial\theta_i} h^*(\xi_i)\right](\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}), \tag{C.9}$$

where ξ_i is an interior point on the line from $\hat{\boldsymbol{\theta}}$ to $\hat{\boldsymbol{\theta}}_N$.
Writing (C.9) in a matrix form gives

$$D^1 h^*(\hat{\boldsymbol{\theta}}_N) = D^1 h^*(\hat{\boldsymbol{\theta}}) + D^2 h^*(\hat{\boldsymbol{\theta}})^T(\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}) + R^*(\xi), \tag{C.10}$$

where $R^*(\xi) = ((\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}})^T[D^2\partial h^*(\xi_1)/\partial\theta_1](\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}), ..., (\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}})[D^2\partial h^*(\xi_d)/\partial\theta_d](\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}}))^T$. Substituting $\hat{\boldsymbol{\theta}}_N - \hat{\boldsymbol{\theta}} = -[D^2h^*(\hat{\boldsymbol{\theta}})]^{-1}D^1h^*(\hat{\boldsymbol{\theta}})$ into (C.10), we have

$$D^1h^*(\hat{\boldsymbol{\theta}}_N) = D^1h^*(\hat{\boldsymbol{\theta}}) - D^2h^*(\hat{\boldsymbol{\theta}})[D^2h^*(\hat{\boldsymbol{\theta}})]^{-1}D^1h^*(\hat{\boldsymbol{\theta}}) + O(n^{-2}) = O(n^{-2}).$$

Proof of Theorem 5.

$$\frac{\int e^{-nh^*(\boldsymbol{\theta})}d\boldsymbol{\theta}}{\int e^{-nh(\boldsymbol{\theta})}d\boldsymbol{\theta}}$$

$$= \left(\frac{|D^2h(\hat{\boldsymbol{\theta}})|}{|D^2h^*(\hat{\boldsymbol{\theta}}_N)|}\right)^{1/2}\frac{C_n^*(\hat{\boldsymbol{\theta}}_N)}{C_n(\hat{\boldsymbol{\theta}})}\frac{exp[-nh^*(\hat{\boldsymbol{\theta}}_N)]}{exp[-nh(\hat{\boldsymbol{\theta}})]}\frac{1+\dfrac{a_{1n}^*}{n}+\dfrac{a_{2n}^*}{n^2}+O(n^{-3})}{1+\dfrac{a_{1n}}{n}+\dfrac{a_{2n}}{n^2}+O(n^{-3})}$$

$$= \left(\frac{|D^2h(\hat{\boldsymbol{\theta}})|}{|D^2h^*(\hat{\boldsymbol{\theta}}_N)|}\right)^{1/2}\frac{C_n^*(\hat{\boldsymbol{\theta}}_N)}{C_n(\hat{\boldsymbol{\theta}})}\frac{exp[-nh^*(\hat{\boldsymbol{\theta}}_N)]}{exp[-nh(\hat{\boldsymbol{\theta}})]}$$
$$\times\left(1+\frac{a_{1n}^*-a_{1n}}{n}+\frac{a_{2n}^*-a_{2n}-a_{1n}(a_{1n}^*-a_{1n})}{n^2}+O(n^{-3})\right)$$

First, we shall evaluate $a_{1n}^* - a_{1n}$. It follows from Lemma 13 (d) that

$$\frac{n^3}{72}\sum_{ijkqrs}h_{ijk}^*(\hat{\boldsymbol{\theta}}_N)h_{qrs}^*(\hat{\boldsymbol{\theta}}_N)\mu_{ijkqrs}^* - \frac{n^3}{72}\sum_{ijkqrs}h_{ijk}(\hat{\boldsymbol{\theta}})h_{qrs}(\hat{\boldsymbol{\theta}})\mu_{ijkqrs} \tag{C.11}$$
$$= \frac{n^3}{72}\sum\Big\{(h_{ijk}^*(\hat{\boldsymbol{\theta}}_N) - h_{ijk}(\hat{\boldsymbol{\theta}}))h_{qrs}^*(\hat{\boldsymbol{\theta}}_N)\mu_{ijkqrs}^*$$
$$+ h_{ijk}(\hat{\boldsymbol{\theta}})(h_{qrs}^*(\hat{\boldsymbol{\theta}}_N) - h_{qrs}(\hat{\boldsymbol{\theta}}))\mu_{ijkqrs}^* + h_{ijk}(\hat{\boldsymbol{\theta}})h_{qrs}(\hat{\boldsymbol{\theta}})(\mu_{ijkqrs}^* - \mu_{ijkqrs})\Big\}$$
$$= \frac{1}{72}\cdot 2\sum\left(\frac{1}{n}\sum h_{\alpha ijk}h^{\alpha\beta}G_\beta - \frac{1}{n}G_{ijk}\right)h_{qrs}[n^3\mu_{ijkqrs}] + \frac{1}{72}\sum h_{ijk}h_{qrs}n^3(\mu_{ijkqrs}^* - \mu_{ijkqrs}) + O(n^{-2})$$

The left term of the foregoing equation becomes

$$\frac{1}{36n}\sum h_{\alpha ijk}h_{qrs}h^{\alpha\beta}[n^3\mu_{ijkqrs}]G_\beta - \frac{1}{36n}\sum h_{qrs}[n^3\mu_{ijkqrs}]G_{ijk}. \tag{C.12}$$

By using Lemma 14 (b), the right term equation is given by

$$\frac{1}{72}\Big\{-\frac{9}{n}\sum h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{9}{n}\sum h_{ijk}h_{qrs}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}G_{\alpha\beta}$$
$$-\frac{18}{n}\sum h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{18}{n}\sum h_{ijk}h_{qrs}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}G_{\alpha\beta}$$
$$-\frac{18}{n}\sum h_{ijk}h_{qrs}h_{l\alpha\beta}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}h^{lm}G_m + \frac{18}{n}\sum h_{ijk}h_{qrs}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}G_{\alpha\beta}\Big\}$$
$$= -\frac{1}{8n}\sum h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{1}{8n}\sum h_{ijk}h_{qrs}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}G_{\alpha\beta} \tag{C.13}$$
$$-\frac{1}{4n}\sum h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}h^{lm}G_m + \frac{1}{4n}\sum h_{ijk}h_{qrs}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}G_{\alpha\beta} \tag{C.14}$$
$$-\frac{1}{4n}\sum h_{ijk}h_{qrs}h_{l\alpha\beta}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}h^{lm}G_m + \frac{1}{4n}\sum h_{ijk}h_{qrs}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}G_{\alpha\beta}. \tag{C.15}$$

In contrast, it follows from Lemma 13 (g), and Lemma 13 (e) that

$$
\begin{aligned}
&-\frac{1}{8}\Big(\sum h^*_{ijkq}(\hat{\boldsymbol\theta}_N)h^{*ij}(\hat{\boldsymbol\theta}_N)h^{*kq}(\hat{\boldsymbol\theta}_N)-\sum h_{ijkq}(\hat{\boldsymbol\theta})h^{ij}(\hat{\boldsymbol\theta})h^{kq}(\hat{\boldsymbol\theta})\Big)\\
&=-\frac{1}{8}\sum_{ijkq}\Big\{h^*_{ijkq}(\hat{\boldsymbol\theta}_N)h^{*ij}(\hat{\boldsymbol\theta}_N)[h^{*kq}(\hat{\boldsymbol\theta}_N)-h^{kq}(\hat{\boldsymbol\theta})]\\
&+h^*_{ijkq}(\hat{\boldsymbol\theta}_N)[h^{*ij}(\hat{\boldsymbol\theta}_N)-h^{ij}(\hat{\boldsymbol\theta})]h^{kq}(\hat{\boldsymbol\theta})+h^{ij}(\hat{\boldsymbol\theta})h^{kq}(\hat{\boldsymbol\theta})[h^*_{ijkq}(\hat{\boldsymbol\theta}_N)-h_{ijkq}(\hat{\boldsymbol\theta})]\Big\}\\
&=-\frac{1}{8}\Big(-\frac{2}{n}\sum h_{ijkq}h^{ij}h^{k\alpha}h_{l\alpha\beta}h^{\beta q}h^{lm}G_m+\frac{2}{n}\sum h_{ijkq}h^{ij}h^{k\alpha}h^{\beta q}G_{\alpha\beta}\\
&+\frac{1}{n}\sum h_{ijkql}h^{ij}h^{kq}h^{lm}G_m-\frac{1}{n}\sum h^{ij}h^{kq}G_{ijkq}\Big)+O(n^{-2})\\
&=\frac{1}{4n}\sum h_{ijkq}h^{ij}h^{k\alpha}h_{l\alpha\beta}h^{\beta q}h^{lm}G_m-\frac{1}{4n}\sum h_{ijkq}h^{ij}h^{k\alpha}h^{\beta q}G_{\alpha\beta}\\
&-\frac{1}{8n}\sum h_{ijkql}h^{ij}h^{kq}h^{lm}G_m+\frac{1}{8n}\sum h^{ij}h^{kq}G_{ijkq}+O(n^{-2}). \qquad \text{(C.16)}
\end{aligned}
$$

Combining (C.12), (C.13), (C.14), (C.15), and (C.16), we have $a^*_{1n}-a_{1n}=\kappa_n/n$, where κ_n is of order 1. Next, we evaluate $a^*_{2n}-a_{2n}$.
Following the proof of Tierney and Kadane (1986), we have

$$
a^*_{2n}-a_{2n}=-\frac{1}{2}\sum_{ijk}h^*_{ijk}(\hat{\boldsymbol\theta}_N)(n^2b^*_i)h^{*jk}+\frac{1}{2}\sum_{ijk}h_{ijk}(\hat{\boldsymbol\theta})(n^2b_i)h^{jk}+O(n^{-1}). \qquad \text{(C.17)}
$$

Therefore,

$$
E[g^+(\boldsymbol\Theta)]=\left(\frac{|D^2h_n(\hat{\boldsymbol\theta})|}{|D^2h^*_n(\hat{\boldsymbol\theta}_N)|}\right)^{1/2}\frac{C^*_n(\hat{\boldsymbol\theta}_N)}{C_n(\hat{\boldsymbol\theta})}\frac{exp[-nh^*(\hat{\boldsymbol\theta}_N)]}{exp[-nh(\hat{\boldsymbol\theta})]}\left(1+\frac{c_n}{n^2}+O(n^{-3})\right),
$$

where $C^*_n(\hat{\boldsymbol\theta}_N)=exp\{(n/2)D^1h^*(\hat{\boldsymbol\theta}_N)^T[D^2h^*(\hat{\boldsymbol\theta}_N)]^{-1}D^1h^*(\hat{\boldsymbol\theta}_N)\}$ and

$$
\begin{aligned}
c_n&=-\frac{n^2}{2}\sum h_{ijk}b^*_ih^{jk}+\frac{n^2}{2}\sum h_{ijk}b_ih^{jk}+\frac{1}{4}\sum h_{ijkq}h_{l\alpha\beta}h^{ij}h^{k\alpha}h^{\beta q}h^{lm}G_m\\
&-\frac{1}{4}\sum h_{ijkq}h^{ij}h^{k\alpha}h^{\beta q}G_{\alpha\beta}-\frac{1}{8}\sum h_{ijkql}h^{ij}h^{kq}h^{lm}G_m+\frac{1}{8}\sum h^{ij}h^{kq}G_{ijkq}\\
&+\frac{1}{36}\sum h_{\alpha ijk}h_{qrs}h^{\alpha\beta}[n^3\mu_{ijkqrs}]G_\beta-\frac{1}{36}\sum h_{qrs}[n^3\mu_{ijkqrs}]G_{ijk}\\
&-\frac{1}{8}\sum_{ijkqrs\alpha\beta lm}h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}h^{lm}G_m+\frac{1}{8}\sum_{ijkqrs\alpha\beta}h_{ijk}h_{qrs}h^{jk}h^{rs}h^{i\alpha}h^{\beta q}G_{\alpha\beta}\\
&-\frac{1}{4}\sum_{ijkqrs\alpha\beta lm}h_{ijk}h_{qrs}h_{l\alpha\beta}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}h^{lm}G_m+\frac{1}{4}\sum_{ijkqrs\alpha\beta}h_{ijk}h_{qrs}h^{jr}h^{ks}h^{i\alpha}h^{\beta q}G_{\alpha\beta}\\
&-\frac{1}{4}\sum_{ijkqrs\alpha\beta lm}h_{ijk}h_{qrs}h_{l\alpha\beta}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}h^{lm}G_m+\frac{1}{4}\sum_{ijkqrs\alpha\beta}h_{ijk}h_{qrs}h^{kq}h^{rs}h^{i\alpha}h^{\beta j}G_{\alpha\beta}
\end{aligned}
$$

As an alternative expression of $-(n^2/2)\sum h_{ijk}b_i^* h^{jk}$ in c_n, from Lemma 14 (d), we have

$$\begin{aligned} b_i^*(\hat{\boldsymbol{\theta}}_N) &= -\sum_l h^{il} h_l^*(\hat{\boldsymbol{\theta}}_N) + O(n^{-3}) \\ &= -\frac{1}{2n^2} \sum_{\alpha\beta lmr} h_{l\alpha\beta} h^{il} h^{\alpha r} h^{\beta m} G_r G_m + O(n^{-3}). \end{aligned}$$

Thus

$$-\frac{n^2}{2} \sum_{ijk} h_{ijk} b_i^* h^{jk} = \frac{1}{4} \sum_{ijklmr\alpha\beta} h_{ijk} h_{l\alpha\beta} h^{il} h^{jk} h^{\alpha r} h^{\beta m} G_r G_m + O(n^{-1}) \tag{C.18}$$

Appendix D: Moments of multivariate normal distributions

A $d \times 1$ random vector $\mathbf{u}$ is according to $N(\mathbf{0}, \Sigma)$, where $\Sigma \equiv (\mu_{ij})$, and $\mathbf{0}$ is a $d \times 1$ null vector. Let μ_{ijkq} and μ_{ijkqrs} denote the fourth and sixth central moments of multivariate normal distribution $N(\mathbf{0}, \Sigma)$. Then we have the formula concerning the moments:

$$\begin{aligned} \mu_{ijkq} &= \mu_{ij}\mu_{ks} + \mu_{ik}\mu_{js} + \mu_{is}\mu_{jk}, \\ \mu_{ijkqrs} &= \mu_{ij}\mu_{kq}\mu_{rs} + \mu_{ij}\mu_{kr}\mu_{qs} + \mu_{ij}\mu_{ks}\mu_{qr} \\ &\quad + \mu_{ik}\mu_{jq}\mu_{rs} + \mu_{ik}\mu_{jr}\mu_{qs} + \mu_{ik}\mu_{js}\mu_{qr} \\ &\quad + \mu_{iq}\mu_{jk}\mu_{rs} + \mu_{iq}\mu_{jr}\mu_{ks} + \mu_{iq}\mu_{js}\mu_{kr} \\ &\quad + \mu_{ir}\mu_{jk}\mu_{qs} + \mu_{ir}\mu_{jq}\mu_{ks} + \mu_{ir}\mu_{js}\mu_{kq} \\ &\quad + \mu_{is}\mu_{jk}\mu_{qr} + \mu_{is}\mu_{jq}\mu_{kr} + \mu_{is}\mu_{jr}\mu_{kq} \end{aligned}$$

Appendix E:Auxiliary calculation on the Laplace approximations

(iii) From Lemma 12 (d),

$$\begin{aligned} &E\left[\frac{n^2}{72}\sum h_{ijk}h_{qrs}z_i z_j z_k z_q z_r z_s\right] \\ &= \frac{n^2}{72}\sum h_{ijk}h_{qrs}E[z_i z_j z_k z_q z_r z_s] \\ &= \frac{n^2}{72}\sum h_{ijk}h_{qrs}\left(\mu_{ijkqrs} + \sum_{[m_1,m_2]_6} b_{m_1}b_{m_2}\mu_{m_3m_4m_5m_6} + \sum_{[l_1,l_2,l_3,l_4]_6} b_{l_1}b_{l_2}b_{l_3}b_{l_4}\mu_{l_5l_6} + b_i b_j b_k b_q b_r b_s\right) \\ &= \frac{n^2}{72}\sum_{ijkqrs} h_{ijk}h_{qrs}\mu_{ijkqrs} + \frac{n^2}{72}\sum_{ijkqrs} h_{ijk}h_{qrs}\left(\sum_{[m_1,m_2]_6} b_{m_1}b_{m_2}\mu_{m_3m_4m_5m_6}\right) + O(n^{-3}) \end{aligned} \tag{E.1}$$

Because

$$\sum_{[m_1,m_2]_6} b_{m_1}b_{m_2}\mu_{m_3m_4m_5m_6} = \underline{b_ib_j\mu_{kqrs}} + \underline{b_ib_k\mu_{jqrs}} + b_ib_q\mu_{jkrs} + b_ib_r\mu_{jkqs} + b_ib_s\mu_{jkqr}$$

$$+ \underline{b_jb_k\mu_{iqrs}} + b_jb_q\mu_{ikrs} + b_jb_r\mu_{ikqs} + b_jb_s\mu_{ikqr} + b_kb_q\mu_{ijrs}$$

$$+ b_kb_r\mu_{ijqs} + b_kb_s\mu_{ijqr} + \underline{b_qb_r\mu_{ijks}} + \underline{b_qb_s\mu_{ijkr}} + \underline{b_rb_s\mu_{ijkq}},$$

$\frac{n^2}{72}\sum h_{ijk}h_{qrs}\Big(\sum_{[m_1,m_2]} b_{m_1}b_{m_2}\mu_{m_3m_4m_5m_6}\Big)$ is devided into two kinds of terms. Hence, the second term in (E.1) is

$$\frac{n^2}{72}\sum h_{ijk}h_{qrs}\Big(\sum_{[m_1,m_2]} b_{m_1}b_{m_2}\mu_{m_3m_4m_5m_6}\Big)$$
$$= \frac{n^2}{72}\cdot 6\sum h_{ijk}h_{qrs}b_ib_j\mu_{kqrs} + \frac{n^2}{72}\cdot 9\sum h_{ijk}h_{qrs}b_ib_q\mu_{jkrs}.$$

Conseqently, we have

$$E\left[\frac{n^2}{72}\sum h_{ijk}h_{qrs}z_iz_jz_kz_qz_rz_s\right]$$
$$= \frac{n^2}{72}\sum h_{ijk}h_{qrs}\mu_{ijkqrs} + \frac{n^2}{12}\sum h_{ijk}h_{qrs}b_ib_j\mu_{kqrs} + \frac{n^2}{8}\sum h_{ijk}h_{qrs}b_ib_q\mu_{jkrs}.$$

(iv) From Lemma 12 (c),

$$E\left[-\frac{n}{120}\sum h_{ijkqr}z_iz_jz_kz_qz_r\right] = -\frac{n}{120}\sum h_{ijkqr}E[z_iz_jz_kz_qz_r]$$
$$= -\frac{n}{120}\sum_{ijkqr} h_{ijkqr}\Big(\sum_{[m_1]_5} b_{m_1}\mu_{m_2m_3m_4m_5}\Big) + O(n^{-3})$$
$$= -\frac{n}{120}\cdot 5\sum_{ijkqr} h_{ijkqr}b_i\mu_{jkqr} + O(n^{-3})$$
$$= -\frac{n}{24}\sum_{ijkqr} h_{ijkqr}b_i\mu_{jkqr} + O(n^{-3}).$$

(v) From Lemma 12 (d)

$$E\left[-\frac{n}{720}\sum h_{ijkqrs}z_iz_jz_kz_qz_rz_s\right] = -\frac{n}{720}\sum h_{ijkqrs}E[z_iz_jz_kz_qz_rz_s]$$
$$= -\frac{n}{720}\sum h_{ijkqrs}\mu_{ijkqrs} + O(n^{-3}).$$

(vi) From Lemma 12 (e),

$$E\left[\frac{n^2}{144}\sum h_{ijk}h_{qrst}z_iz_jz_kz_qz_rz_sz_t\right]$$
$$= \frac{n^2}{144}\sum h_{ijk}h_{qrst}E[z_iz_jz_kz_qz_rz_sz_t]$$
$$= \frac{n^2}{144}\sum h_{ijk}h_{qrst}\Big(\sum_{[m_1]_7} b_{m_1}\mu_{m_2m_3m_4m_5m_6m_7}\Big) + O(n^{-3})$$
$$= \frac{n^2}{144}\cdot 3\sum h_{ijk}h_{qrst}b_i\mu_{jkqrst} + \frac{n^2}{144}\cdot 4\sum h_{ijk}h_{qrst}b_q\mu_{ijkrst} + O(n^{-3})$$
$$= \frac{n^2}{48}\sum h_{ijk}h_{qrst}b_i\mu_{jkqrst} + \frac{n^2}{36}\sum h_{ijk}h_{qrst}b_q\mu_{ijkrst} + O(n^{-3}).$$

(vii) From Lemma 12 (f),

$$E\left[\frac{n^2}{1152}\sum h_{ijkq}h_{rstu}z_iz_jz_kz_qz_rz_sz_tz_u\right] = \frac{n^2}{1152}\sum h_{ijkq}h_{rstu}E[z_iz_jz_kz_qz_rz_sz_tz_u]$$
$$= \frac{n^2}{1152}\sum h_{ijkq}h_{rstu}\mu_{ijkqrstu} + O(n^{-3}).$$

(viii) From Lemma 12 (f),

$$E\left[\frac{n^2}{720}\sum h_{ijk}h_{qrstu}z_iz_jz_kz_qz_rz_sz_tz_u\right] = \frac{n^2}{720}\sum h_{ijk}h_{qrstu}\mu_{ijkqrstu} + O(n^{-3}).$$

(ix) From Lemma 12 (g),

$$E\left[-\frac{n^3}{1296}\sum h_{ijk}h_{qrs}h_{tuv}z_iz_jz_kz_qz_rz_sz_tz_uz_v\right]$$
$$= -\frac{n^3}{1296}\sum h_{ijk}h_{qrs}h_{tuv}\left(\sum_{[m_1]_9} b_{m_1}\mu_{m_2m_3m_4m_5m_6m_7m_8m_9}\right) + O(n^{-3})$$
$$= -\frac{n^3}{1296}\cdot 9\sum h_{ijk}h_{qrs}h_{tuv}b_i\mu_{jkqrstuv} + O(n^{-3})$$
$$= -\frac{n^3}{144}\sum h_{ijk}h_{qrs}h_{tuv}b_i\mu_{jkqrstuv} + O(n^{-3}).$$

(x) From Lemma 12 (h),

$$E\left[-\frac{n^3}{1728}\sum h_{ijk}h_{qrs}h_{tuvw}z_iz_jz_kz_qz_rz_sz_tz_uz_vz_w\right] = -\frac{n^3}{1728}\sum h_{ijk}h_{qrs}h_{tuvw}\mu_{ijkqrstuvw} + O(n^{-3})$$

(xi) From Lemma 12 (i),

$$E\left[\frac{n^4}{31104}\sum h_{ijk}h_{qrs}h_{tuv}h_{wxy}z_iz_jz_kz_qz_rz_sz_tz_uz_vz_wz_xz_y\right]$$
$$= \frac{n^4}{31104}\sum h_{ijk}h_{qrs}h_{tuv}h_{wxy}\mu_{ijkqrstuvwxy} + O(n^{-3}).$$

The inequality in Theorem 5.

$$\frac{1+\dfrac{a^*_{1n}}{n}+\dfrac{a^*_{2n}}{n^2}+\dfrac{a^*_{3n}}{n^3}}{1+\dfrac{a_{1n}}{n}+\dfrac{a_{2n}}{n^2}+\dfrac{a_{3n}}{n^3}} = 1 + \frac{a^*_{1n}-a_{1n}}{n} + \frac{a^*_{2n}-a_{2n}-a_{1n}(a^*_{1n}-a_{1n})}{n^2} + O(n^{-3}),$$

where $O(n^{-3})$ is given by

$$\left[\frac{a^*_{3n}}{n^3} - \frac{a_{3n}}{n^3} - \left(\frac{a_{1n}}{n}+\frac{a_{2n}}{n^2}+\frac{a_{3n}}{n^3}\right)\left(\frac{a^*_{2n}-a_{2n}-a_{1n}(a^*_{1n}-a_{1n})}{n^2}\right) - \left(\frac{a_{2n}}{n^2}+\frac{a_{3n}}{n^3}\right)\left(\frac{a^*_{1n}-a_{1n}}{n}\right)\right]$$
$$\times \frac{1}{1+\dfrac{a_{1n}}{n}+\dfrac{a_{2n}}{n^2}+\dfrac{a_{3n}}{n^3}}$$

PROOF.

$$\frac{1+\dfrac{a_{1n}^*}{n}+\dfrac{a_{2n}^*}{n^2}+\dfrac{a_{3n}^*}{n^3}}{1+\dfrac{a_{1n}}{n}+\dfrac{a_{2n}}{n^2}+\dfrac{a_{3n}}{n^3}}-\left(1+\frac{a_{1n}^*-a_{1n}}{n}+\frac{a_{2n}^*-a_{2n}-a_{1n}(a_{1n}^*-a_{1n})}{n^2}\right)$$

$$=\left[1+\frac{a_{1n}^*}{n}+\frac{a_{2n}^*}{n^2}+\frac{a_{3n}^*}{n^3}-\left(1+\frac{a_{1n}^*-a_{1n}}{n}+\frac{a_{2n}^*-a_{2n}-a_{1n}(a_{1n}^*-a_{1n})}{n^2}\right)\left(1+\frac{a_{1n}}{n}+\frac{a_{2n}}{n^2}+\frac{a_{3n}}{n^3}\right)\right]$$

$$\times\frac{1}{1+\dfrac{a_{1n}}{n}+\dfrac{a_{2n}}{n^2}+\dfrac{a_{3n}}{n^3}}$$

Therefore this inequality is proved.

References

Harville, D. A. (1997), *Matrix Algebra From A Statisticians Perspective*, Springer-Verlag, New York.

Kass, R. E., Tierney, L., and Kadane, J. B. (1990), "The Validity of Posterior Expansions Based on Laplace's Method,"in *Essays in Honor of George Barnard*, eds. S. Geisser, J. S. Hodges, S. J. Press, and A. Zellner, Amsterdam: NorthHolland.

Laplace, P. S. (1986), "Memoir on the Probability of Causes of Events" (translation by S. Stiger of the 1774 original), *Statistical Science*, 1, 364-378.

Lindley, D. V. (1980), "Approximate Bayesian Methods," in *Bayesian Statistics*, eds. J. M. Bernardo, M. H. Degroot, D. V. Lindley, and A. F. M. Smith, Valencia, Spain: University Press, pp. 223-245.

Miyata, Y., (2004), "Fully Exponential Laplace Approximations Using Asymptotic Modes," *Journal of the American Statistical Association*, 99, 1037-1049.

Mosteller, F., and Wallace, D. L. (1964), *Inference and Disputed Authorship: The Federalist Papers*, Reading, MA: Addison-Wesley.

Tierney, L., and Kadane, J. B. (1986), "Accurate Approximations for Posterior Moments and Marginal Densities," *Journal of the American Statistical Association*, 81, 82-86.