[5] $X \sim B(n,p)$.

① $g_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$

$= (pe^t + 1 - p)^n$

② $\frac{d}{dt} g_X(t)\Big|_{t=0} = n(pe^t+1-p)^{n-1} \cdot pe^t \Big|_{t=0} = np = E(X)$.

$E(X^2) = \frac{d^2}{dt^2} g_X(t)\Big|_{t=0} = n(n-1)(pe^t+1-p)^{n-2} p^2 e^{2t} + n(pe^t+1-p)^{n-1} pe^t \Big|_{t=0}$

$= n(n-1)p^2 + np$.

$V(X) = E(X^2) - E(X)^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np(1-p)$ //

[6] $Z = \frac{X-\mu}{\sigma} \sim N(0,1^2)$ より

$Y = Z^2 = \frac{(X-\mu)^2}{\sigma^2} \sim \chi^2_1$ //

← くわしくは P83 例 5.5-3 に書いてある

[7] $Cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-E(X))(y-E(Y)) f_{XY}(x,y)\,dx\,dy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy - yE(X) - xE(Y) + E(X)E(Y)) f_{XY}(x,y)\,dx\,dy$

$= E(XY) - E(X)\int_{-\infty}^{\infty} y \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}_{= f_Y(y)}\,dy - E(Y)\int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}_{= f_X(x)}\,dx + E(X)E(Y)$

$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$

$= E(XY) - E(X)E(Y)$ //

② $X \perp\!\!\!\perp Y$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$= E(X)E(Y) - E(X)E(Y) = 0$.

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[8] $f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(x-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right\}\right\}$

$\underset{0}{\rho}$ $\underset{0}{\rho}$ $\underset{0}{\rho}$

$= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}$

$= \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right\}$

$= f_X(x)\,f_Y(y).$ $\quad\therefore X \perp\!\!\!\perp Y$

[9] $\sin x = \sin 0 + (\sin x)'|_{x=0}\,x + (\sin x)''|_{x=0}\frac{x^2}{2} + (\sin x)'''|_{x=\delta}\frac{x^3}{6}$

$(\sin x)' = \cos x$

$(\sin x)'' = (\cos x)' = -\sin x$

$(\sin x)''' = (-\sin x)' = -\cos x$

$\therefore \sin x = x - (\cos\delta)\cdot\frac{x^3}{6}$

[10]

① $E(\bar{X}) = \mu,\ V(\bar{X}) = \frac{\sigma^2}{n}$ より $E((\bar{X})^2) = V(\bar{X}) + E(\bar{X})^2 = \frac{\sigma^2}{n} + \mu^2$

② $E(X_i^2) = \sigma^2 + \mu^2$

$E\left[\frac{1}{n}\sum_{i=1}^{n}X_i^2\right] = \frac{1}{n}\sum_{i=1}^{n}E(X_i^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2) = \sigma^2+\mu^2$

③ $V(S^2) = E\left[\frac{1}{n}\sum_{i=1}^{n}X_i^2\right] - E\left[(\bar{X})^2\right] = \sigma^2+\mu^2 - \frac{\sigma^2}{n} - \mu^2 = \frac{n-1}{n}\sigma^2$

[11]

① $E[\overline{X^2}] = E\left[\frac{1}{n}\sum_{i=1}^{n}X_i^2\right] = \frac{1}{n}\sum_{i=1}^{n}E[X_i^2] = \mu_2'$

② $V[\overline{X^2}] = V\left[\frac{1}{n}\sum_{i=1}^{n}X_i^2\right] = \frac{1}{n^2}\sum_{i=1}^{n}V[X_i^2] = \frac{A}{n}$

③ $P(|\overline{X^2} - \mu_2'| > \varepsilon) \leq \frac{E[(\overline{X^2}-\mu_2')^2]}{\varepsilon^2} = \frac{A}{\varepsilon^2 n} \to 0\ (n\to\infty)$