最初に言うこと．・最後にまとめてやることが難しい科目
・なるべく休まないこと
・演習で用いた問題は後で出てくるので必ず解くこと．

No.

Date

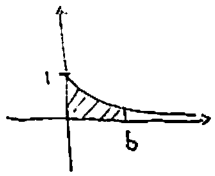
数理統計学

・4年生がいる場合の就活対応．
・来週の金曜日に Waseda-net で流す

[無限積分]　$\int_a^{\infty} f(x)dx \overset{\text{def}}{=} \lim_{b\to+\infty} \int_a^b f(x)dx$.

定義する

$y=e^{-x}$



$$\int_0^b e^{-x}dx = [-e^{-x}]_0^b = -e^{-b}+1 \longrightarrow 1 \quad (b\to\infty)$$

$$\int_0^{\infty} e^{-x}dx = 1.$$

注　実際には　$\int_0^{\infty} e^{-x}dx = [-e^{-x}]_0^{\infty} = 0+1=1$　← 途中で∞がでてこなければOK

[2重積分]：定義 + イメージ → ppt or pdf.
計算方法(累次積分)

$D=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le 2\}$　　$f(x,y)=\sqrt{x+y}$

$$\iint_D f(x,y)dxdy = \int_0^2\left(\int_0^1 \sqrt{x+y}\,dx\right)dy$$

POINT
$\boxed{\text{yを定数とみなす}}$

$$= \int_0^2 \left[\frac{2}{3}(x+y)^{3/2}\right]_0^1 dy$$

$$= \int_0^2 \left(\frac{2}{3}(1+y)^{3/2} - \frac{2}{3}y^{3/2}\right)dy$$

$$= \frac{2}{3}\left[\frac{2}{5}(1+y)^{5/2} - \frac{2}{5}y^{5/2}\right]_0^2$$

$$= \frac{4}{15}(9\sqrt{3} - 4\sqrt{2} - 1)$$

・一般の定義域における定積分．

・時間か

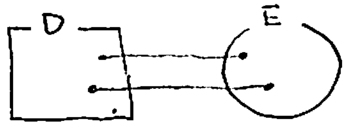
・CLTのイメージ

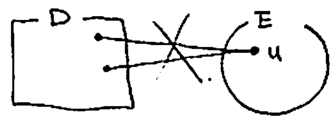
[変数変換](置換積分の重積分版)

$I = \iint_D f(x,y)\,dx\,dy$

・$h(x,y) = \begin{pmatrix} h_1(x,y) \\ h_2(x,y) \end{pmatrix}$ をDからEへの1対1変換

$\begin{cases} u = h_1(x,y) \\ v = h_2(x,y) \end{cases} \Leftrightarrow \begin{array}{l} x = g_1(u,v) \\ y = g_2(u,v) \end{array}$





$$J(u,v) = \begin{vmatrix} \dfrac{\partial g_1(u,v)}{\partial u} & \dfrac{\partial g_1(u,v)}{\partial v} \\ \dfrac{\partial g_2(u,v)}{\partial u} & \dfrac{\partial g_2(u,v)}{\partial v} \end{vmatrix}$$

をヤコビアンと言う。

ここは絶対値

$$\iint_D f(x,y)\,dx\,dy = \iint_E f(g_1(u,v), g_2(u,v))\,|J(u,v)|\,du\,dv$$

ex. $D = \{(x,y) \mid 1 \le x+y \le 2,\ 0 \le x-y \le 1\}$

・$\iint_D \dfrac{x-y}{x+y}\,dx\,dy$

$\begin{cases} u = \underbrace{x+y}_{h_1(x,y)} \\ v = \underbrace{x-y}_{h_2(x,y)} \end{cases} \Leftrightarrow \begin{cases} x = \underbrace{\dfrac{u+v}{2}}_{g_1(u,v)} \\ y = \dfrac{u-v}{2} \end{cases}$ $\quad E = \{(u,v) \mid 1 \le u \le 2,\ 0 \le v \le 1\}$

$\dfrac{\partial g_1(u,v)}{\partial u} = \dfrac{1}{2}$ $\quad \dfrac{\partial g_1(u,v)}{\partial v} = \dfrac{1}{2}$, $\quad \dfrac{\partial g_2(u,v)}{\partial u} = \dfrac{1}{2}$, $\quad \dfrac{\partial g_2(u,v)}{\partial v} = -\dfrac{1}{2}$

$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\dfrac{1}{2}$

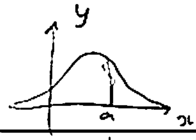
$$\iint_D \frac{x-y}{x+y}\,dx\,dy = \int_0^1 \int_1^2 \frac{v}{u} \underbrace{\frac{1}{2}}_{|J|}\,du\,dv = \int_0^1 \frac{v}{2}\left(\int_1^2 \frac{1}{u}\,du\right)dv$$

$$= \int_0^1 \frac{v}{2}\,[\log u]_1^2\,dv$$

$$= \int_0^1 \frac{\log 2}{2}\,v\,dv = \frac{\log 2}{4}.$$

数理統計学

$\int_a^b f(x)dx - \int_a^a f(x)dx$

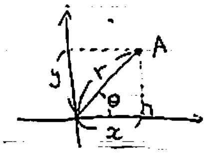


$0 < u \leq 2$

Date . . No.

$\int_a^b f(x)dx \quad \int_a^a f(x)dx = 0$

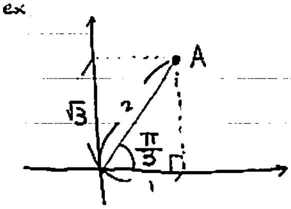
【極座標】



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Leftrightarrow \quad r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}$$

$A(r,\theta)$ を A の極座標と言う.

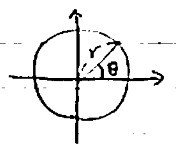
ex



直交座標 $A(1,\sqrt{3})$

極座標 $A(2,\frac{\pi}{3})$

【極座標を用いた変数変換】



$$\iint_{x^2+y^2\leq 1} \sqrt{x^2+y^2}\, dx\, dy.$$

$x = r\cos\theta$
$y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$= \iint_{r^2\leq 1} \underbrace{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}_{r}\; r\, dr\, d\theta$$

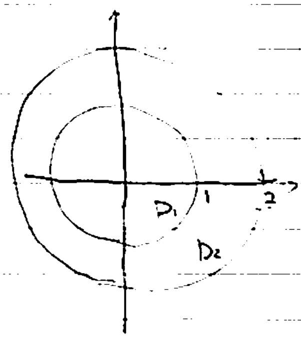
$$= \iint_{\substack{0\leq r\leq 1 \\ 0\leq\theta<2\pi}} r^2\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2 dr\, d\theta = \left(\int_0^1 r^2 dr\right)\left(\int_0^{2\pi} d\theta\right) = \left[\frac{1}{3}r^3\right]_0^1 [\theta]_0^{2\pi} = \frac{1}{3}\cdot 2\pi = \frac{2\pi}{3}$$

[重積分における無限積分]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2+y^2}{2}\right\} dx\,dy = \lim_{n\to\infty} \iint_{D_n} \exp\left\{-\frac{x^2+y^2}{2}\right\} dx\,dy.$$

$\longleftarrow \exp(\square) = e^{\square}$



例えば $D_n = \{(x,y) \mid x^2+y^2 \leq n^2\}$.

$$= \lim_{n\to\infty} \iint_{x^2+y^2\leq n^2} \exp\left\{-\frac{x^2+y^2}{2}\right\} dx\,dy$$

$x = r\cos\theta$
$y = r\sin\theta$ $\quad |J| = r$.

$D_1 \subseteq D_2 \subseteq \cdots$ だんだん大きくなる

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_0^{n} \exp\left\{-\frac{\overbrace{r^2\cos^2\theta + r^2\sin^2\theta}^{r^2}}{2}\right\} r\,dr\,d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left[-\exp\left(-\frac{r^2}{2}\right)\right]_0^n d\theta \qquad \int_0^{2\pi} d\theta = 2\pi$$

$$= \lim_{n\to\infty} 2\pi\left(1-\exp\left(-\frac{n^2}{2}\right)\right) = 2\pi.$$

実際には、∞は1つの数として扱う.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dx\,dy = \int_0^{2\pi}\int_0^{\infty} r e^{-\frac{r^2}{2}} dr = 2\pi\left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} = 2\pi$$

$D = \{(x,y) \mid -\infty < x < \infty,\ -\infty < y < \infty\}$
$E = \{(r,\theta) \mid 0 \leq r < \infty,\ 0 \leq \theta < 2\pi\}$

$I = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$

$$I^2 = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\, \exp\left(-\frac{y^2}{2}\right) dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}-\frac{y^2}{2}\right) dx\,dy = 2\pi.$$

ここでは説明しない

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx = \sqrt{2\pi}$$

[ガンマ関数] $s>0$ に対して $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$ を<u>ガンマ関数</u>と言う.

- $\Gamma(1) = 1$.
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(s+1) = s\,\Gamma(s)$

$$\left(\because\ \Gamma(s+1) = \int_0^{\infty} x^s e^{-x} dx = \left[x^s(-e^{-x})\right]_0^{\infty} - \int_0^{\infty} s x^{s-1}(-e^{-x}) dx = s\int_0^{\infty} x^{s-1} e^{-x} dx = s\,\Gamma(s)\right)$$

数理統計学

ex.1 サイコロを1回振る.

X:出た目

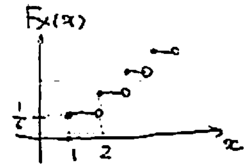
X	1	2	3	4	5	6	計
確率	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

‖
P と略.　　$\mathcal{X}=\{1,2,3,4,5,6\}$.

$F_X(1)=P(X\le 1)=\frac{1}{6}$

$F_X(2)=P(X\le 2)=\frac{2}{6}$

⋮



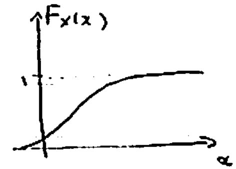
$f_X(2)=F_X(2)-F_X(2-0)$

$=\frac{2}{6}-\frac{1}{6}=\frac{1}{6}$

ex2. ある地点での気温を測る

X:気温

$P(25\le X<26)=\int_{25}^{26}f_X(x)dx$



p33.

確率変数

離散型

(X)	x_1	x_2	…	x_k	計
P	P_1	P_2	…	P_k	1

[確率分布]

‖
$P(X=x_2)$

kは∞でもok

$f_X(x_i)=P(X=x_i)\quad (i=1,2,\dots,k)$

を確率量関数 (Probability Mass Function) p.m.f.

[標本空間]: Xの値域を標本空間と言う

[累積分布関数] $F_X(x_i)=P(X\le x_i)\quad (i=1,\dots,k)$.

Cumulative Distribution Function c.d.f.

$f_X(c)=F_X(c)-F_X(c-0)$

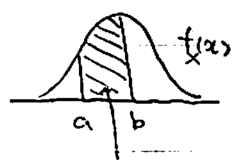
ここで $F_X(c-0)=\lim_{\varepsilon\downarrow 0}F_X(c-\varepsilon)$

[期待値] $E[X]=\sum_{i=1}^{k}x_i f_X(x_i)$

確率分布の中心を表わす

$E[g(X)]=\sum_{i=1}^{k}g(x_i)f_X(x_i)$

連続型



$P(a\le X\le b)=\int_a^b f_X(x)dx$

$\int_{-\infty}^{\infty}f_X(x)dx=1$

$f_X(x)$を確率密度関数 (Probability Density Function) p.d.f.

$F_X(x)=\int_{-\infty}^{x}f_X(t)dt$

$f_X(x)=\frac{d}{dx}F(x)$ ← $\frac{d}{dx}\int_{-\infty}^{x}f(t)dx=f(x)$

$E[X]=\int_{-\infty}^{\infty}x f_X(x)dx$

$E[g(X)]=\int_{-\infty}^{\infty}g(x)f_X(x)dx$

[分散] $V[X]=\sum_{i=1}^{k}(x_i-E[X])^2 f_X(x_i)$ $\qquad$ $V[X]=\int_{-\infty}^{\infty}(x-E[X])^2 f_X(x)\,dx$

確率分布の散らばり具合を表わす.

P38 (i)〜(iv) が成立つ.

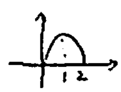
[分散の計算公式] $V[X]=E[X^2]-E[X]^2$ 〃

標準偏差: $D(X)=\sqrt{V(X)}$ 〃

原点周りのr次の積率 (モーメント) $E(X^r)=\mu'_r$ 〃

平均 〃 $E[(X-\mu)^r]=\mu_r$ 〃

ex2. p.d.f. $f(x)=\frac{3}{4}x(2-x)$ $(0\leq x\leq 2)$



$$E[X]=\int_0^2 x\cdot\frac{3}{4}x(2-x)\,dx=1$$

途中式は略

$$\mu'_2=E[X^2]=\int_0^2 x^2\cdot\frac{3}{4}x(2-x)\,dx=\frac{6}{5}$$

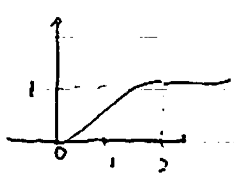
$$\mu_2=V[X]=E[X^2]-E[X]^2=\frac{6}{5}-1^2=\frac{1}{5}$$

$$D(X)=\sqrt{\frac{1}{5}}=\frac{1}{\sqrt{5}}$$

$$F(x)=\int_0^x\frac{3}{4}t(2-t)\,dt=\frac{3}{4}\int_0^x(2t-t^2)\,dt$$

$$=\frac{3}{4}\left[t^2-\frac{1}{3}t^3\right]_0^x$$

$$=\frac{3}{4}\left(x^2-\frac{x^3}{3}\right)\quad(0\leq x\leq 2)$$



$F(x)=1$ $\quad(x>2)$.

典型的な離散型分布

[二項分布]: Xが以下の p.m.f. を持つ時, Xは2項分布$B(n,p)$に従うと言う. ($X \sim B(n,p)$ の略記)

$$f_X(x) = \binom{n}{x} p^x \underbrace{(1-p)^{n-x}}_{q} \qquad \begin{pmatrix} x=0,1,\dots,n \\ 0<p<1 \end{pmatrix}$$

ここで $\binom{n}{x} = {}_nC_x$.　　グラフはP41. 図4.3.1

$$\mu = E(X) = np$$
$$\mu_2 = V(X) = npq$$
$$\mu_3 = E[(X-\mu)^3] = npq(q-p).$$

ex. 表が出る確率が$\frac{2}{5}$である歪んだコインを4回振る

X: 表の出た回数

$X \sim B(4, \frac{2}{5})$　　$f_X(x) = \binom{4}{x} (\frac{2}{5})^x (\frac{3}{5})^{4-x}$　　$x=0,1,\dots,4$

PROOF:

$$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p \cdot p^{x-1} (1-p)^{n-x} \qquad n-x = n-1-(x-1)$$

$$= np \sum_{k=0}^{n} \frac{(n-1)!}{k!(n-1-k)!} p^k (1-p)^{n-1-k}$$

$$= np \, (\underbrace{p + (1-p)}_{1})^{n-1}$$

$$= np$$

[ポアソン分布] Xが以下のp.m.f.を持つ時，Xはポアソン分布$Po(\lambda)$に従うと言い $X \sim Po(\lambda)$と書く．

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad (x=0,1,\ldots\ldots).$$

$\mu = E[X] = \lambda$

$\mu_2 = V[X] = \lambda$

$\mu_3 = E[(X-\mu)^3]$

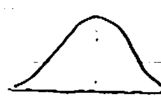
ex ある交差点における交通事故数

[正規分布]：Xが以下のp.d.f.を持つ時，Xは正規分布$N(\mu,\sigma^2)$に従うと言い $X \sim N(\mu,\sigma^2)$と書く．

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \qquad -\infty < x < \infty$$

$\leftarrow \exp\{\square\} = e^{\square}$

パラメーター：$-\infty < \mu < \infty$，$\sigma > 0$



μ

① $E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx = \mu$

② $\mu_2 = V[X] = \sigma^2$

③ $\mu_3 = E[(X-\mu)^3] = 0$

④ $\mu_4 = E[(X-\mu)^4] = 3\sigma^4$

特に$N(0,1^2)$を標準正規分布と言う

p.d.f. $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

c.d.f. $\Phi(x) = \int_{-\infty}^{x}\phi(t)dt$

⑤ $Y = aX+b \sim N(a\mu+b,\ b^2\sigma^2)$

- $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$

PROOF:

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\, dx\, dy$$

$$= \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(x^2+y^2)\right\} dx\, dy}_{= 2\pi} = 1$$

ここは演習でやるので おそらく不用

$I = 1$ □.

PROOF OF ①

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$z = \frac{x-\mu}{\sigma}$ とおく

$\sigma\, dz = dx$

$$= \int_{-\infty}^{\infty} (\mu + z\sigma) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz$$

$$= \mu + \sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz$$

$$= \mu + \sigma \left[\frac{-1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \right]_{-\infty}^{\infty} = \mu.$$

$$V[X] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \sigma^2 z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

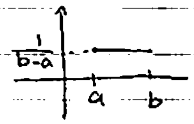
$$= \sigma^2 \underbrace{\int_{-\infty}^{\infty} -z \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)' dz}_{= \underbrace{\left[-z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right]_{-\infty}^{\infty}}_{=0} + \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}_{=1}}$$

$= \sigma^2$

ex. 身長や実験誤差の分布

[一様分布]: Xが以下のp.d.fを持つ時, Xは一様分布 $U(b,a)$ に従うと言い $X \sim U(a,b)$ と書く

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & (a \le x \le b) \\ 0 & \text{その他} \end{cases}$$



① $\mu = E[X] = \dfrac{a+b}{2}$

② $\mu_2 = V[X] = \dfrac{(b-a)^2}{12}$

③ $\mu_3 = 0$

[カイ2乗分布] P195

Xが以下のp.d.fを持つ時, Xは自由度nのカイ2乗分布に従うと言い $X \sim \chi_n^2$ と書く

$$f_X(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} & (x > 0) \\ 0 & (x \le 0) \end{cases}$$

① $\mu = E(X) = n$

② $\mu_2 = V(X) = 2n$

③ $\mu_3 = 8n$

ここで $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ はガンマ関数と呼ばれる

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$\Gamma(1) = 1$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

定理

$$\boxed{Z \sim N(0,1^2) \Rightarrow Y = Z^2 \sim \chi_1^2}$$

PROOF: $P(Y \le y) = P(Z^2 \le y)$

$= P(-\sqrt{y} \le Z \le \sqrt{y})$

$= \int_{-\sqrt{y}}^{\sqrt{y}} \phi(z)dz$

$= 2\int_0^{\sqrt{y}} \phi(z)dz$

$$f_Y(y) = \frac{\partial}{\partial y}P(Y \le y) = 2\phi(\sqrt{y})(\sqrt{y})' = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2}\frac{1}{\sqrt{y}} = \frac{1}{2^{\frac{1}{2}}\sqrt{\pi}} \cdot y^{\frac{1}{2}-1} e^{-y}$$ □