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Utility as a Function of Consumption, Asset and Their Changes

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Abstract

Yonemoto (2013) has characterized the behavior of an individual whose instantaneous utility is a function of the change in his/her consumption, as well as the absolute level. This study extends its model, generalizing the objective function and considering several specific applications including utility as a function of the second-order change, the rate of the (first-order) change and the level of the remaining asset.

Further, for several *given* consumption paths, the corresponding lifetime utility levels are compared. Their order differs depending on the functional form.

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1. Introduction

Most macroeconomic models, such as the Discounted Utility Model, assume that utility of a household or an individual is a function of consumption in each period.[1] However, "Easterlin Paradox," originally studied by Easterlin (1974, updated in 1995) and followed by many others, points out that the level of "happiness" of an individual does not seem to be determined solely by the absolute level of income or consumption.[2]

One way to interpret the paradox is the "Relative Income Hypothesis," introduced by Duesenberry (1949) and focused on the role of comparison with others or one's own state in the past in determining the level of happiness.

Models of habit formation have been also developed in a similar line of thought. Among them are Pollak (1970), which is one of the earliest studies, Gilboa (1989), which takes into account utility variation between two consecutive periods and Loewenstein and Prelec (1993), which considers global properties of a sequence.

As for macroeconomic analyses, Bordley (1986) and Frank (1989) formulate continuous-time models. Carroll et al. (1997) and Carroll et al. (2000) explicitly model the speed of adaptation. Wakai (2008) and Wakai (2013) present models that

[1] Among the earliest Discounted Utility Models is the one introduced by Samuelson (1937).

[2] Recent developments in this field are summarized by Clark et al. (2008).

characterize “utility smoothing” based on the experimental study of Loewenstein (1987).

Noting those studies, Yonemoto (2013) has investigated the behavior and welfare of an individual whose instantaneous utility is a function of the change in his/her consumption, as well as the absolute level. It has shown that, an individual tends to seek an upward-sloping consumption path but if there are initial and terminal conditions, the overall shape of the path depends highly on them.

This study extends Yonemoto (2013) and considers more general formulation. Moreover, several specific cases are explored using simple models.

In the next section, the level of lifetime satisfaction of an individual is expressed by a generalized function of sequences of consumption and asset. The corresponding optimal conditions are derived.

In Section 3, a simplified model, which is based on that of Yonemoto (2013), is presented and the case in which the second-order difference matters is investigated.

Section 4 and 5 explore the cases in which the rate of (first-order) change and the level of the remaining asset are taken into account, respectively.

Section 6 compares the levels of lifetime satisfaction that correspond to several typical (given) consumption paths. Some examples of the reversal in their order, depending on the functional form, are presented.

2. Maximization Problem

First, consider the general case in which the instantaneous utility of an individual is a function of the first and higher-order derivatives of his/her consumption and asset with respect to time t as well as their absolute levels. Denote the level of his/her consumption at time t by $c_t^{(0)}$ and its n-th order derivative by $c_t^{(n)}$. Similarly, denote the level of his/her asset by $a_t^{(0)}$ and its m-th order derivative by $a_t^{(m)}$. Assume that an individual lives for $0 \le t \le T$ and asset function Φ evaluates $a_T^{(0)}$, the level of asset remaining at T. Then, the intertemporal utility (lifetime satisfaction) maximization problem with any constraint $g(\cdot)$ is formulated as follows:[3]

$$\max_{c^{(N)}} \int_0^T u\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right) dt + \Phi\left(a_T^{(0)}\right)$$

subject to:

$$\frac{dc_t^{(n-1)}}{dt} = c_t^{(n)} \quad \text{for} \quad n = 1, \cdots, N,$$

$$\frac{da_t^{(m-1)}}{dt} = a_t^{(m)} \quad \text{for} \quad m = 1, \cdots, M, \quad \text{and}$$

$$g\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right) = 0 .$$

$$c_0^{(n)} = \overline{c}_0^{(n)} \quad \text{and} \quad c_T^{(n)} = \overline{c}_T^{(n)} \quad \text{for} \quad n = 0, \cdots, N-1 \quad \text{and}$$

$$a_0^{(m)} = \overline{a}_0^{(m)} \quad \text{for} \quad m = 0, \cdots, M . \tag{1}$$

[3] Any realistic setting may have non-negative constraint on c: $c_t^{(0)} \ge 0$.

Associated optimal conditions are summarized in the Appendix. However, the solutions of this type of problem can be discontinuous (e.g. bang-bang) and not always characterized adequately by the continuous model. Thus, consider also the following discrete version:

$$\sum_{t=1}^{T} u\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right) + \Phi\left(a_T^{(0)}\right)$$

subject to

$$g\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right) = 0 \quad \text{for} \quad t = 1, \cdots, T\ ,$$

$$c_0^{(0)} = \overline{c}_0^{(0)}, \cdots, c_0^{(N-1)} = \overline{c}_0^{(N-1)},$$

$$c_T^{(0)} = \overline{c}_T^{(0)},$$

$$a_0^{(0)} = \overline{a}_0^{(0)}, \cdots, a_0^{(M-1)} = \overline{a}_0^{(M-1)},$$

where

$$c_t^{(n)} \equiv c_t^{(n-1)} - c_{t-1}^{(n-1)} \quad \text{for} \quad t = 1, \cdots, T \quad \text{and} \quad n = 1, \cdots, N\ ,$$

$$a_t^{(m)} \equiv a_t^{(m-1)} - a_{t-1}^{(m-1)} \quad \text{for} \quad t = 1, \cdots, T \quad \text{and} \quad m = 1, \cdots, M\ . \tag{2}$$

Note that there are several ways to approximate the derivatives by discrete variables. [4] (2) simply uses the difference of the current value (of the lower-order variable) from the previous one. [5]

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[4] Other definitions may include the difference between the next and current values (e.g. $c_t^{(n)} \equiv c_{t+1}^{(n-1)} - c_t^{(n-1)}$,) or more generally, any weighted average of the differences around t (e.g. $c_t^{(n)} \equiv \alpha\left(c_t^{(n-1)} - c_{t-1}^{(n-1)}\right) + (1-\alpha)\left(c_{t+1}^{(n-1)} - c_t^{(n-1)}\right)$.)

[5] This simple approximation may cause some deviations from the continuous case. However, it is consistent with the settings of habit formation models. In those models, the utility depends on the

The Lagrangian for the constrained maximization problem (2) is:

$$L=\sum_{t=1}^{T}u\left(t,c_t^{(0)},c_t^{(1)},\cdots,c_t^{(N)},a_t^{(0)},a_t^{(1)},\cdots,a_t^{(M)}\right)+\Phi\left(a_T^{(0)}\right)$$
$$+\sum_{t=1}^{T}\sum_{n=1}^{N}\lambda_t^{c(n)}\left\{\left(c_t^{(n-1)}-c_{t-1}^{(n-1)}\right)-c_t^{(n)}\right\}+\sum_{t=1}^{T}\sum_{m=1}^{M}\lambda_t^{a(m)}\left\{\left(a_t^{(m-1)}-a_{t-1}^{(m-1)}\right)-a_t^{(m)}\right\}$$
$$+\sum_{t=1}^{T}\mu_t g\left(t,c_t^{(0)},c_t^{(1)},\cdots,c_t^{(N)},a_t^{(0)},a_t^{(1)},\cdots,a_t^{(M)}\right) \tag{3}$$

Differentiating (3) with respect to the control variables, first-order conditions are obtained as follows:

For $t=1,\cdots,T-1$,

i-1) $$\frac{\partial L^*}{\partial c_t^{(0)}}=\frac{\partial u^*}{\partial c_t^{(0)}}+\lambda_t^{c(1)*}-\lambda_{t+1}^{c(1)*}+\mu_t\frac{\partial g^*}{\partial c_t^{(0)}}=0, \tag{4}$$

$$\frac{\partial L^*}{\partial c_t^{(n)}}=\frac{\partial u^*}{\partial c_t^{(n)}}-\lambda_t^{c(n)*}+\lambda_t^{c(n+1)*}-\lambda_{t+1}^{c(n+1)*}+\mu_t\frac{\partial g^*}{\partial c_t^{(n)}}=0 \tag{5}$$

for $n=1,\cdots,N-1$,

$$\frac{\partial L^*}{\partial c_t^{(N)}}=\frac{\partial u^*}{\partial c_t^{(N)}}-\lambda_t^{c(N)*}+\mu_t\frac{\partial g^*}{\partial c_t^{(N)}}=0, \tag{6}$$

i-2) $$\frac{\partial L^*}{\partial c_T^{(n)}}=\frac{\partial u^*}{\partial c_T^{(n)}}-\lambda_T^{c(n)*}+\lambda_T^{c(n+1)*}+\mu_T\frac{\partial g^*}{\partial c_T^{(n)}}=0, \tag{7}$$

for $n=1,\cdots,N-1$,

$$\frac{\partial L^*}{\partial c_T^{(N)}}=\frac{\partial u^*}{\partial c_T^{(N)}}-\lambda_T^{c(N)*}+\mu_T\frac{\partial g^*}{\partial c_T^{(N)}}=0, \tag{8}$$

For $t=1,\cdots,T-1$,

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difference of the level of the consumption from "reference point." If the point is updated every period, the setting coincides with that of (2) at first-order level.

ii-1) $$\frac{\partial L^*}{\partial a_t^{(0)}} = \frac{\partial u^*}{\partial a_t^{(0)}} + \lambda_t^{a(1)*} - \lambda_{t+1}^{a(1)*} + \mu_t \frac{\partial g^*}{\partial a_t^{(0)}} = 0, \tag{9}$$

$$\frac{\partial L^*}{\partial a_t^{(m)}} = \frac{\partial u^*}{\partial a_t^{(m)}} - \lambda_t^{a(m)*} + \lambda_t^{a(m+1)*} - \lambda_{t+1}^{a(m+1)*} + \mu_t \frac{\partial g^*}{\partial a_t^{(m)}} = 0, \tag{10}$$

for $\quad m = 1, \cdots, M-1,$

$$\frac{\partial L^*}{\partial a_t^{(M)}} = \frac{\partial u^*}{\partial a_t^{(M)}} - \lambda_t^{a(M)*} + \mu_t \frac{\partial g^*}{\partial a_t^{(M)}} = 0, \tag{11}$$

ii-2) $$\frac{\partial L^*}{\partial a_T^{(0)}} = \frac{\partial u^*}{\partial a_T^{(0)}} + \frac{\partial \Phi^*}{\partial a_T^{(0)}} + \lambda_T^{a(1)*} + \mu_T \frac{\partial g^*}{\partial a_T^{(0)}} = 0, \tag{12}$$

$$\frac{\partial L^*}{\partial a_T^{(m)}} = \frac{\partial u^*}{\partial a_T^{(m)}} - \lambda_T^{a(m)*} + \lambda_T^{a(m+1)*} + \mu_T \frac{\partial g^*}{\partial a_T^{(m)}} = 0 \quad \text{for} \quad m = 1, \cdots, M-1, \tag{13}$$

$$\frac{\partial L^*}{\partial a_T^{(M)}} = \frac{\partial u^*}{\partial a_T^{(M)}} - \lambda_T^{a(M)*} + \mu_T \frac{\partial g^*}{\partial a_T^{(M)}} = 0, \tag{14}$$

Consider the following type of constraint, which is consistent with the specification in and after section 3:

$$g\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right) = g\left(c_t^{(0)}, a_t^{(0)}, a_t^{(1)}\right) \tag{15}$$

Also, assume that each partial derivative is constant:

$$\frac{\partial g}{\partial c_t^{(0)}} = g_{c0}, \quad \frac{\partial g}{\partial a_t^{(0)}} = g_{a0}, \quad \frac{\partial g}{\partial a_t^{(1)}} = g_{a1}. \tag{16}$$

Then, by i-1) and ii-1), the following relationship is derived:

$$\frac{g_{a0} + g_{a1}}{g_{c0}} \left[\frac{\partial u^*}{\partial c_t^{(0)}} + \sum_{w=1}^{N} \sum_{z=1}^{2^w} \left\{ (-1)^{Q_{wz}} u_{c,t}^{wz} \right\} \right] - \frac{g_{a1}}{g_{c0}} \left[\frac{\partial u^*}{\partial c_{t+1}^{(0)}} + \sum_{w=1}^{N} \sum_{z=1}^{2^w} \left\{ (-1)^{Q_{wz}} u_{c,t+1}^{wz} \right\} \right]$$

$$- \left[\frac{\partial u^*}{\partial a_t^{(0)}} + \sum_{w=1}^{M} \sum_{z=1}^{2^w} \left\{ (-1)^{Q_{wz}} u_{a,t}^{wz} \right\} \right]$$

$$= 0\,, \tag{17}$$

where

$$Q_{wz} = \sum_{i=0}^{w-1} q_i^{wz}\ ,$$

$$\left(q_0^{wz}, \cdots, q_{w-1}^{wz}\right) = \left\{ \left(q_0, \cdots, q_{w-1}\right) \middle| z = \sum_{i=0}^{w-1} q_i\, 2^i, q_i = 0{,}1 \right\} \quad \text{and}$$

$$u_{c,t}^{wz} = \frac{\partial u}{\partial c_{t+Q_{wz}}^{(w)}}\,.$$

(17) is a kind of Euler's equation, which does not always have an explicit solution.[6]

[6] For the properties of Euler's equation with higher-order derivatives, see Chiang (2000).

3. Second-Order Difference

Next, characterize the behavior of an individual who cares about the second-order difference in consumption, as well as the first-order difference and the absolute level, using a simplified model. Suppose that the instantaneous utility of an individual is additively separable and expressed by:

$$\tilde{u}_t = \alpha_0 \left(c_t^{(0)}\right)^{\beta_0} + \alpha_1 \left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1} + \alpha_2 \left(c_t^{(2)} + \tilde{c}^{(2)}\right)^{\beta_2} . \tag{18}$$

where $\tilde{c}^{(1)},\ \tilde{c}^{(2)} > 0$.[7]

Then, the intertemporal utility maximization problem is written as:

$$\sum_{t=1}^{T} \left[\left(\frac{1}{1+\rho} \right)^{t-1} \left\{ \alpha_0 \left(c_t^{(0)}\right)^{\beta_0} + \alpha_1 \left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1} + \alpha_2 \left(c_t^{(2)} + \tilde{c}^{(2)}\right)^{\beta_2} \right\} \right], \tag{19}$$

$$a_t^{(0)} = (1+r)\left(a_{t-1}^{(0)} - c_t^{(0)}\right),$$

$$c_0^{(0)} = \bar{c}_0^{(0)}, \quad c_0^{(1)} = \bar{c}_0^{(1)}, \quad c_T^{(0)} = \bar{c}_T^{(0)}, \quad a_0^{(0)} = \bar{a}_0^{(0)} \quad \text{and}$$

$$c_t^{(1)} \equiv c_t^{(0)} - c_{t-1}^{(0)},$$

where r and ρ are the interest rate and discount rate, respectively.

The structural equations are combined into a single equation by adding up:

$$a_0^{(0)} = \sum_{t=1}^{T} \left\{ \left(\frac{1}{1+r} \right)^{t-1} c_t^{(0)} \right\} + \left(\frac{1}{1+r} \right)^{T} a_T^{(0)} . \tag{20}$$

The Lagrangian for (19) is, taking into account (20),

[7] In the second and third terms, sufficiently large $\tilde{c}^{(1)}$ and $\tilde{c}^{(2)}$ are added, respectively, in order to guarantee positive values in the parentheses.

$$L = \sum_{t=1}^{T}\left[\left(\frac{1}{1+\rho}\right)^{t-1}\left\{\alpha_0\left(c_t^{(0)}\right)^{\beta_0} + \alpha_1\left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1} + \alpha_2\left(c_t^{(2)} + \tilde{c}^{(2)}\right)^{\beta_2}\right\}\right]$$

$$+ \sum_{t=1}^{T}\lambda_t^{c(1)}\left\{\left(c_t^{(0)} - c_{t-1}^{(0)}\right) - c_t^{(1)}\right\} + \sum_{t=1}^{T}\lambda_t^{c(2)}\left\{\left(c_t^{(1)} - c_{t-1}^{(1)}\right) - c_t^{(2)}\right\}$$

$$+ \mu\left\{a_0^{(0)} - \sum_{t=1}^{T}\left\{\left(\frac{1}{1+r}\right)^{t-1} c_t^{(0)}\right\} - \left(\frac{1}{1+r}\right)^{T-1} a_T^{(0)}\right\}. \qquad (21)$$

Differentiating (21) with respect to the control variables $c_1^{(0)}, \cdots, c_{T-1}^{(0)}$, $c_1^{(1)}, \cdots, c_T^{(1)}$ and $c_1^{(2)}, \cdots, c_T^{(2)}$, the following first-order conditions are derived:

$$\frac{\partial L^*}{\partial c_t^{(0)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_0 \beta_0 \left(c_t^{(0)}\right)^{\beta_0 - 1} + \lambda_t^{c(1)} - \lambda_{t+1}^{c(1)} - \mu\left(\frac{1}{1+r}\right)^{t-1} = 0 \ ,$$

$$\text{for} \quad t = 1, \cdots, T-1, \qquad (22)$$

$$\frac{\partial L^*}{\partial c_t^{(1)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_1 \beta_1 \left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1 - 1} - \lambda_t^{c(1)} + \lambda_t^{c(2)} - \lambda_{t+1}^{c(2)} = 0 .$$

$$\text{for} \quad t = 1, \cdots, T-1, \qquad (23)$$

$$\frac{\partial L^*}{\partial c_T^{(1)}} = \left(\frac{1}{1+\rho}\right)^{T-1} \alpha_1 \beta_1 \left(c_T^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1 - 1} - \lambda_T^{c(1)} + \lambda_T^{c(2)} = 0 . \qquad (24)$$

$$\frac{\partial L^*}{\partial c_t^{(2)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_2 \beta_2 \left(c_t^{(2)} + \tilde{c}^{(2)}\right)^{\beta_2 - 1} - \lambda_t^{c(2)} = 0, \quad \text{for } t = 1, \cdots, T\ , \qquad (25)$$

Without Taking into Account the Second-Order Change

Before proceeding further with the above setting, consider the case in which the second-order-difference does not matter ($\alpha_2 = 0$) and $\lambda_t^{c(2)} = 0$ by (25). Yonemoto (2013) has analyzed the case in detail. Main conclusions are as follows:

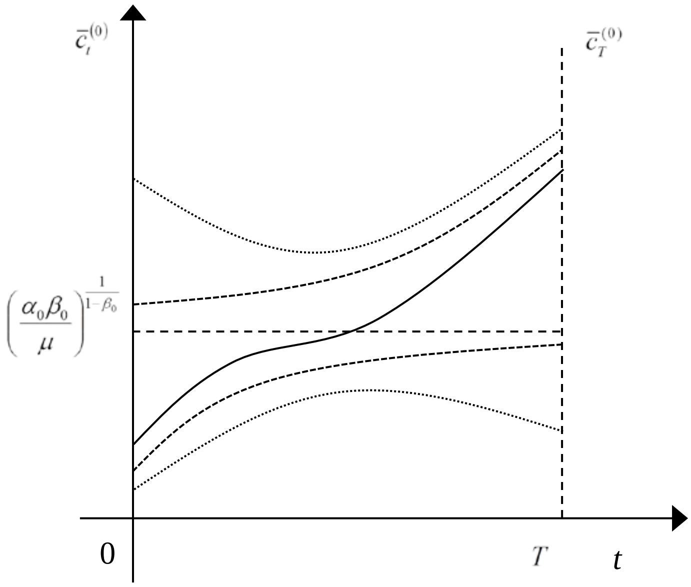

Figure 3.1 Optimal Paths for $\alpha_0, \alpha_1 > 0$, Derived in Yonemoto(2013)

Such as in Figure 3.1, the optimal path is drawn so that it smoothly connects the start point with the end point, while keeping some level of consumption as long as possible. It is concave when the amount of initial asset is relatively large, convex when it is relatively small and has both concave and convex parts when the amount is in between them.

Taking into Account the Second-Order Change

Now, consider the case in which the second-order difference in consumption is taken into account. Assume $\alpha_0 = \alpha_1 = 0$ (and $r = \rho = 0$) at first so that only the second-order matters. Then, (23) is rewritten to be,

$$\lambda_{t+1}^{c(2)} - \lambda_t^{c(2)} = -\lambda_t^{c(1)} \qquad \text{for} \quad t = 1, \cdots, T-1. \tag{26}$$

By (24) and (25),

$$\lambda_T^{c(1)} = \lambda_T^{c(2)} = \alpha_2 \beta_2 \left(c_T^{(2)} + \tilde{c}^{(2)}\right)^{\beta_2 - 1} > 0. \tag{27}$$

By (22) and (27),

$$\lambda_1^{c(1)} \geq \lambda_2^{c(1)} \geq \cdots \geq \lambda_T^{c(1)} > 0, \tag{28}$$

where the equalities hold for $\mu = 0$.

Thus, by (25) and (26),

$$c_1^{(2)} \leq c_2^{(2)} \leq \cdots \leq c_T^{(2)} \tag{29}$$

That is, optimal $c_t^{(1)}$ is non-convex in time.

Typical paths of $c_t^{(1)}$ and $c_t^{(0)}$ are illustrated in Figure 3.2 and 3.3, respectively.

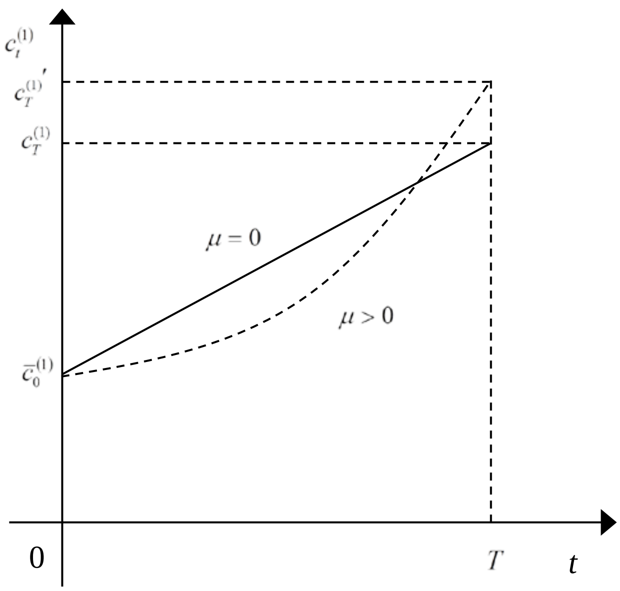

Figure 3.2 Optimal Paths of $c_t^{(1)}$ ($\alpha_0 = \alpha_1 = 0$ and $r = \rho = 0$)

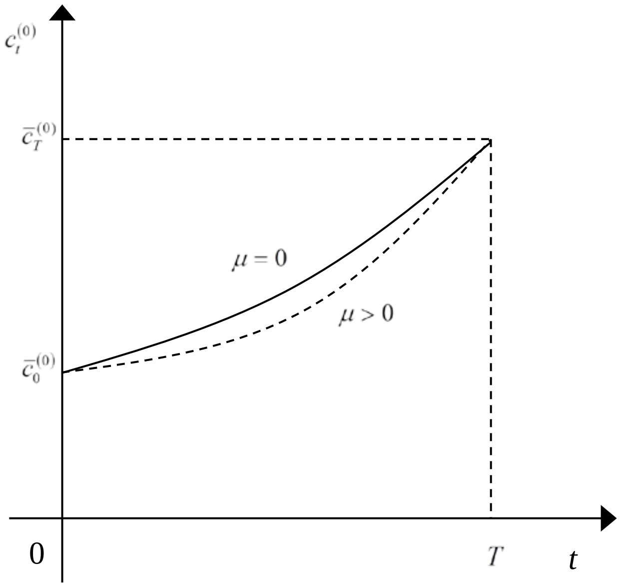

Figure 3.3 Optimal Paths of $c_t^{(0)}$ ($\alpha_0 = \alpha_1 = 0$ and $r = \rho = 0$)

It is more difficult to generally characterize the cases in which $\alpha_0, \alpha_1 > 0$. Instead of listing all possible cases, this study shows a numerical solution as an example. The solid line of Figure 3.4 depicts the optimal path of $c_t^{(0)}$ for the case where $T = 7$, $r = \rho = 0$, $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = 0.5$, $\beta_0 = \beta_1 = \beta_2 = 0.5$, $\tilde{c}^{(1)} = 10$, $\tilde{c}^{(2)} = 50$, $c_0^{(0)} = c_0^{(1)} = 0$, $c_7^{(0)} = 307.72$ and $a_0^{(0)} = 1000$ while the broken line corresponds to the case where $\alpha_2 = 0$ (and all others are the same.)[8]

When the second-order difference is taken into account, the optimal path is drawn so as to reduce the concavity (where it exists) and increase the convexity. In

[8] To illustrate the most "natural" case, $c_7^{(0)} = 307.72$, the optimal value when it can be controlled (and $a_2 = 0$,) is used as a terminal condition. The same value is also used for $a_2 > 0$ to make a comparison.

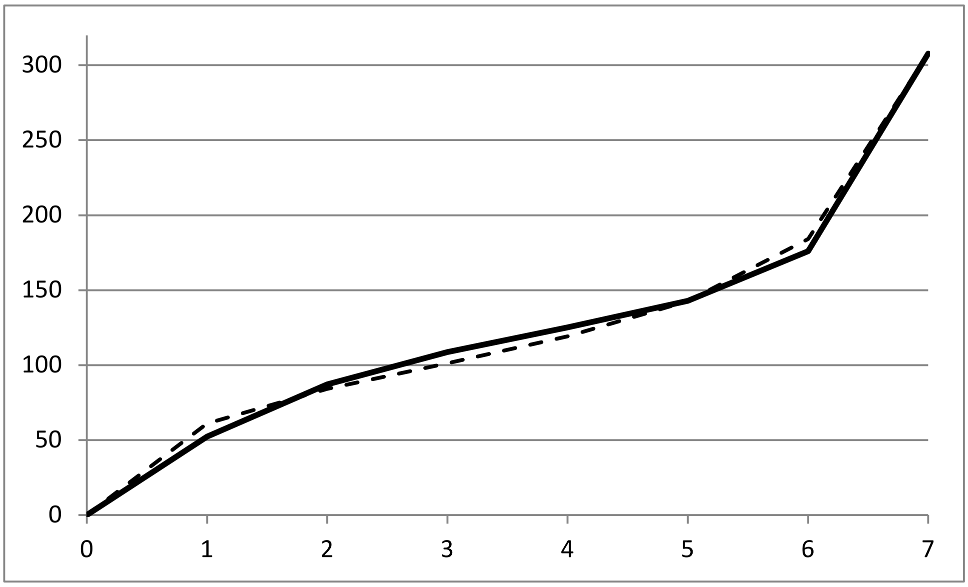

Figure 3.4 Optimal Path of $c_t^{(0)}$ (numerical example)

the cases such as in Figure 3.4, consumption is delayed where concavity or convexity is high, given initial and terminal conditions.

4. Valuing the Rate of the Change

In this section, consider another variation: utility is a function of the *rate* of the first-order change (i.e. growth rate) in consumption. Many psychological works, such as Weber-Fechner law in Fechner (1860), have argued that sensation of human is determined by the relative magnitudes of stimuli but not their absolute levels. Carroll et al. (1997) and Carroll et al. (2000) use a model that describes habit formation by the ratio of the current level to some reference, instead of the difference.

Assume that instantaneous utility is written as follows: [9]

$$\tilde{u}_t = \alpha_0 \left(c_t^{(0)}\right)^{\beta_0} + \alpha_1 \left(c_t^{(1)}\right)^{\beta_1} \tag{30}$$

where $c_t^{(1)} \equiv \dfrac{c_t^{(0)}}{c_{t-1}^{(0)}}$.

The Lagrangian is,

$$\begin{aligned} L = &\sum_{t=1}^{T} \left[\left(\frac{1}{1+\rho} \right)^{t-1} \left\{ \alpha_0 \left(c_t^{(0)}\right)^{\beta_0} + \alpha_1 \left(c_t^{(1)}\right)^{\beta_1} \right\} \right] \\ &+ \sum_{t=1}^{T} \lambda_t \left\{ \frac{c_t^{(0)}}{c_{t-1}^{(0)}} - c_t^{(1)} \right\} \\ &+ \mu \left\{ a_0 - \sum_{t=1}^{T} \left\{ \left(\frac{1}{1+r} \right)^{t-1} c_t^{(0)} \right\} - \left(\frac{1}{1+r} \right)^{T} a_T \right\}. \end{aligned} \tag{31}$$

Differentiating (31) with respect to the control variables, the first-order conditions are obtained as follows:

[9] $\tilde{c}^{(1)}$ is not necessary here because $c_t^{(1)}$ is nonnegative.

$$\frac{\partial L^*}{\partial c_t^{(0)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_0 \beta_0 \left(c_t^{(0)}\right)^{\beta_0 - 1} + \lambda_t \frac{1}{c_{t-1}^{(0)}} - \lambda_{t+1} \frac{c_{t+1}^{(0)}}{\left(c_t^{(0)}\right)^2} - \mu \left(\frac{1}{1+r}\right)^{t-1} = 0 \ ,$$

$$\text{for} \quad t = 1, \cdots, T-1, \qquad (32)$$

$$\frac{\partial L^*}{\partial c_t^{(1)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_1 \beta_1 \left(c_t^{(1)}\right)^{\beta_1 - 1} - \lambda_t = 0, \qquad \text{for} \quad t = 1, \cdots, T \ , \qquad (33)$$

Substituting λ_t and λ_{t+1} of (33) into (32), the following expression is obtained:

$$\left(\frac{1}{1+\rho}\right)^{t-1} \alpha_0 \beta_0 \left(c_t^{(0)}\right)^{\beta_0 - 1} + \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_1 \beta_1 \left(c_t^{(1)}\right)^{\beta_1 - 1} \frac{1}{c_{t-1}^{(0)}}$$

$$- \left(\frac{1}{1+\rho}\right)^{t} \alpha_1 \beta_1 \left(c_{t+1}^{(1)}\right)^{\beta_1 - 1} \frac{c_{t+1}^{(0)}}{\left(c_t^{(0)}\right)^2} - \mu \left(\frac{1}{1+r}\right)^{t-1} = 0 \quad \text{for} \quad t = 1, \cdots, T-1. \qquad (34)$$

Only the Rate of the Change Matters

First, consider the simplest case in which $\alpha_0 = 0$ and $r = \rho = 0$. In addition, if initial asset $a_0^{(0)}$ is large enough to make $\mu = 0$, it follows from (34) and the definition of $c_t^{(1)}$ that $c_{t+1}^{(1)} = c_t^{(1)}$. Thus, in this simplest case, the rate of the change must be constant. Figure 4.1 indicates the corresponding consumption path. Note that, when the rate is constant, logarithmic transformation makes the line to be straight.

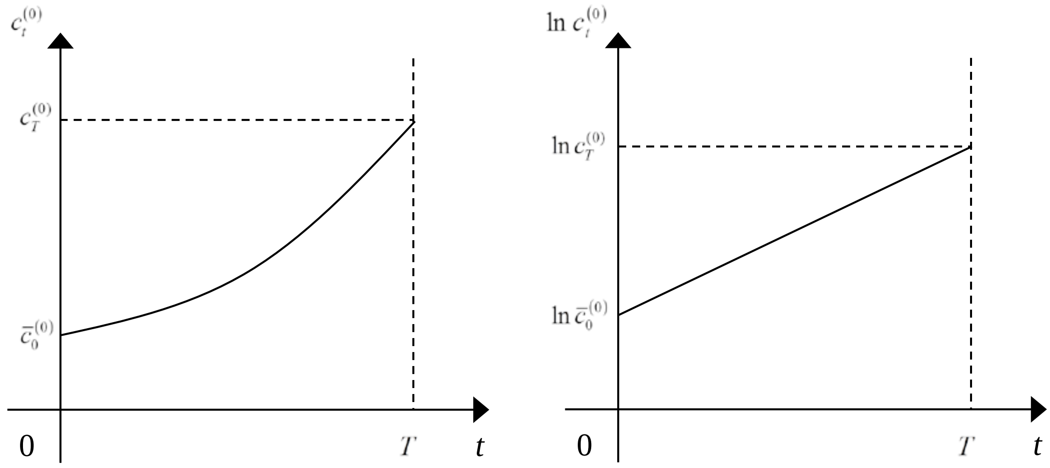

Figure 4.1 Only the Rate of the Change Matters (Left: Original, Right: In Terms of Logarithms)

Both the Rate of the Change and the Absolute Level Matter

Now, consider more general case in which $\alpha_0, \mu > 0$ while $r = \rho = 0$. Then, (34) is rewritten as:

$$-\alpha_1\beta_1\left(c_t^{(1)}\right)^{\beta_1} + \alpha_1\beta_1\left(c_{t+1}^{(1)}\right)^{\beta_1} = \alpha_0\beta_0\left(c_t^{(0)}\right)^{\beta_0} - \mu c_t^{(0)} \tag{35}$$

Thus,

$$c_{t+1}^{(1)} \begin{matrix} > \\ = \\ < \end{matrix} c_t^{(1)} \quad \text{as} \quad c_t^{(0)} \begin{matrix} > \\ = \\ < \end{matrix} \left(\frac{\alpha_0\beta_0}{\mu}\right)^{\frac{1}{1-\beta_0}}. \tag{36}$$

That is, the convexity is larger for higher $c_t^{(0)}$ and smaller for lower $c_t^{(0)}$. For increasing $c_t^{(0)}$, typical patterns are drawn such as in Figure 4.2. Logarithmic version well characterizes the behavior the curves: When a curve is convex (concave,) the

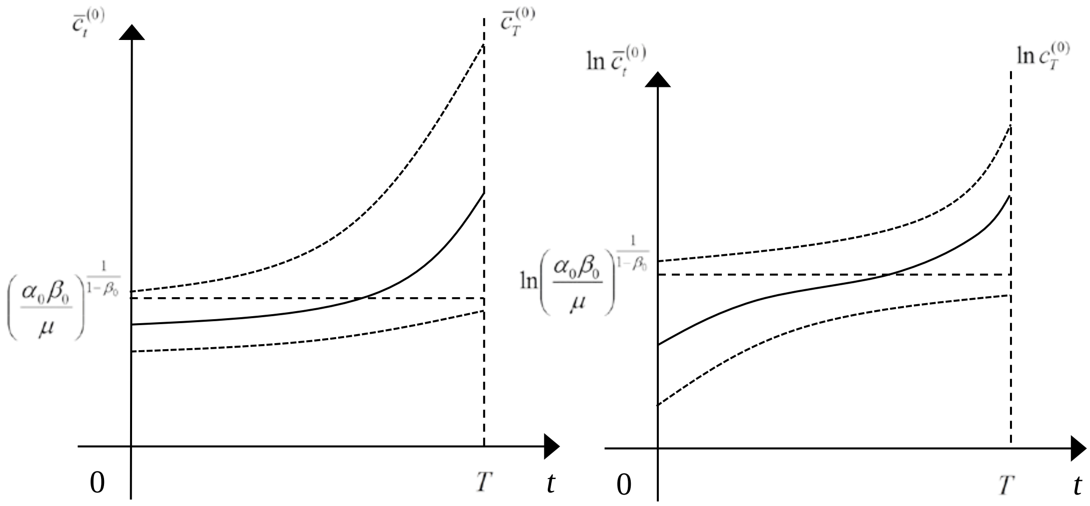

Figure 4.2 The Cases in which α_0, $\mu > 0$ (Left: Original, Right: In Terms of Logarithms)

growth rate is increasing (decreasing.) If the amount of initial asset is small so that μ is large, an individual consumes relatively small amount at first and the growth rate increases at an increasing rate.

5. Valuing Asset in Each Period

As the third variation of the model, consider the case in which utility is also a function of the level of asset remaining in each period. Suppose, instead of (18) or (30), the instantaneous utility is expressed as follows:

$$\tilde{u}_t = \alpha_a \left(a_t^{(0)}\right)^{\beta_a} + \alpha_0 \left(c_t^{(0)}\right)^{\beta_0} + \alpha_1 \left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1}, \tag{37}$$

As for the initial and terminal conditions, use the ones in (19).

$$c_0^{(0)} = \bar{c}_0^{(0)}, \quad c_0^{(1)} = \bar{c}_0^{(1)}, \quad c_T^{(0)} = \bar{c}_T^{(0)} \quad \text{and} \quad a_0^{(0)} = \bar{a}_0^{(0)}.$$

Then, the Lagrangian is rewritten to be:

$$\begin{aligned} L = &\sum_{t=1}^{T} \left[\left(\frac{1}{1+\rho}\right)^{t-1} \left\{ \alpha_a \left(a_t^{(0)}\right)^{\beta_a} + \alpha_0 \left(c_t^{(0)}\right)^{\beta_0} + \alpha_1 \left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1} \right\} \right] \\ &+ \sum_{t=1}^{T} \lambda_t \left\{ \left(c_t^{(0)} - c_{t-1}^{(0)}\right) - c_t^{(1)} \right\} \\ &+ \sum_{t=1}^{T} \mu_t \left\{ (1+r)\left(a_{t-1}^{(0)} - c_t^{(0)}\right) - a_t^{(0)} \right\}. \end{aligned} \tag{38}$$

The first-order conditions are as follows:

$$\frac{\partial L^*}{\partial a_t^{(0)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_a \beta_a \left(a_t^{(0)}\right)^{\beta_a - 1} - \mu_t + (1+r)\mu_{t+1} = 0 \quad \text{for} \quad t = 1, \cdots, T, \tag{39}$$

$$\frac{\partial L^*}{\partial c_t^{(0)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_0 \beta_0 \left(c_t^{(0)}\right)^{\beta_0 - 1} + \lambda_t - \lambda_{t+1} - \mu_t = 0 \quad \text{for} \quad t = 1, \cdots, T-1 \tag{40}$$

$$\frac{\partial L^*}{\partial c_t^{(1)}} = \left(\frac{1}{1+\rho}\right)^{t-1} \alpha_1 \beta_1 \left(c_t^{(1)} + \tilde{c}^{(1)}\right)^{\beta_1 - 1} - \lambda_t = 0, \quad \text{for} \quad t = 1, \cdots, T. \tag{41}$$

Only Asset Matters

First of all, if only the level of asset matters ($\alpha_0 = \alpha_1 = 0$), it follows directly that $c_t^{(0)} = 0$ for all periods. That is, an individual keeps the initial asset $\overline{a}_0^{(0)}$ for life because consumption does not affect his/her utility at all.

Both Asset and Consumption Matter

Next, consider the case where $\alpha_a, \alpha_0 > 0$ but $\alpha_1 = 0$. For $r = \rho = 0$, $\mu_{t+1} < \mu_t$ by (39) so that $c_{t+1}^{(0)} > c_t^{(0)}$. The profile is typically drawn as in Figure 5.1. An individual postpone consumption in order not to reduce his/her asset in early periods.[10]

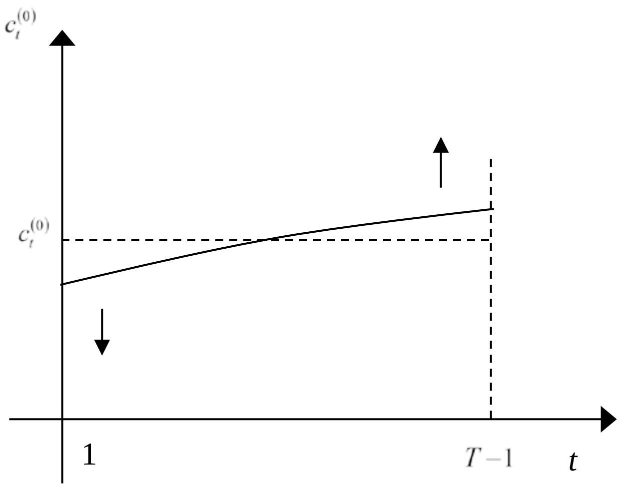

Figure 5.1 Valuing Asset

[10] $c_t^{(0)}$ $(1 \le t \le T-1)$ is determined independently from $\overline{c}_0^{(0)}$ or $\overline{c}_T^{(0)}$ as long as $\alpha_1 = 0$.

Asset, Consumption and Its Change Matter

Finally, assume that $\alpha_a, \alpha_0, \alpha_1 > 0$ (and $r = \rho = 0$.)

Again, $\mu_{t+1} < \mu_t$ by (39). By (40), either $c_t^{(0)}$ is smaller or $\lambda_t - \lambda_{t+1}$ is larger (or both) in earlier periods. Although the results depend on the parameters in general, noting $\bar{c}_0^{(0)}$ and $\bar{c}_T^{(0)}$ affect λ_1 and λ_T, respectively, the profile is typically drawn such as in Figure 5.2. The solid line corresponds to the case where the asset matters ($\alpha_a > 0$,) in contrast to the the broken one ($\alpha_a = 0$.)

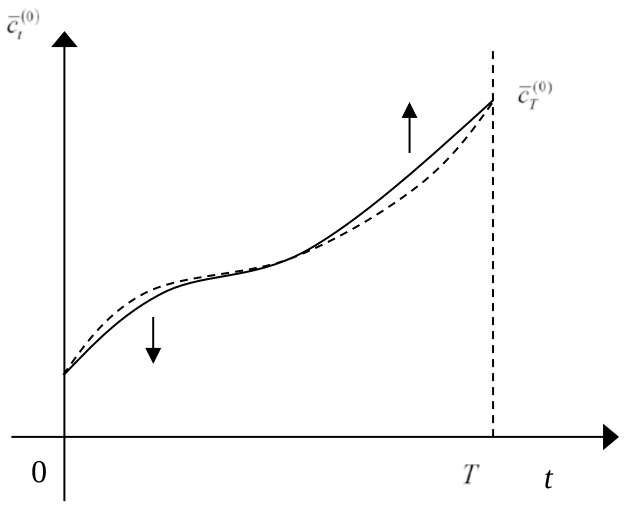

Figure 5.2 Typical Path When Asset, Consumption and Its Change are Valued

6. Evaluating Actual Growth

So far, the study has dealt with the *optimal* consumption path at a viewpoint of initial period. However, because of any liquidity constraints or uncertainty about future income, sometimes the consumption of an individual can be largely affected by his/her current income, which is given exogenously by the society. This section compares the lifetime utility levels that correspond to several typical consumption paths to each other, assuming an individual does not control them. First, start with the cases in which only the absolute level of consumption or its first-order change matters.

6.1 Absolute Value or First-Order (Rises and Declines)

Consider five typical consumption paths, indicated in Figure 6.1, which characterize increases and decreases in consumption. The relative orders of the lifetime utility levels are summarized in the table below, assuming $\rho = 0$. If $\alpha_0 > 0$ and $\alpha_1 = \alpha_2 = 0$ in the model of (18), only the absolute level of consumption matters so that generally the higher the consumption path is, the better. Moreover, if $\beta_0 = 1$, marginal utility of consumption does not diminishes. That is, whenever he/she consumes, his/her lifetime satisfaction rises by the same amount. As a result, paths i, ii and iii are equivalent (although most experimental or empirical studies show that few people actually feel so.) When $\alpha_0 = \alpha_2 = 0$ but $\alpha_1 > 0$, an individual is only concerned about the *change*. As a result, ii and iv (iii and v) are equivalent while the resources necessary for those paths are different. Note that, while v is better than i or ii for an individual who values the absolute level (and v requires more resources,) it is worse than i or ii for one who values the change.

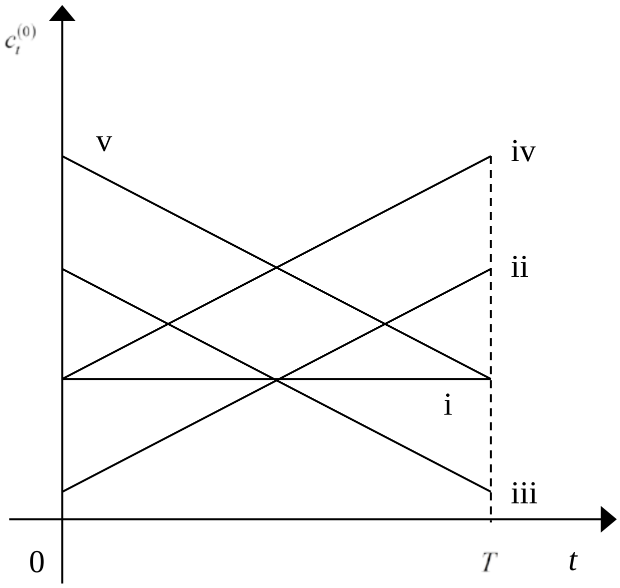

Figure 6.1 Rises and Declines

$\alpha_0 > 0$, $\alpha_1 = \alpha_2 = 0$ and $\beta_0 = 1$:

$$U^*_{iv} = U^*_v > U^*_i = U^*_{ii} = U^*_{iii}$$

$\alpha_0 > 0$, $\alpha_1 = \alpha_2 = 0$ and $\beta_0 < 1$:

$$U^*_{iv} = U^*_v > U^*_i > U^*_{ii} = U^*_{iii}$$

$\alpha_0 = \alpha_2 = 0$ and $\alpha_1 > 0$:

$$U^*_{ii} = U^*_{iv} > U^*_i > U^*_{iii} = U^*_v$$

6.2 Soar, Plunge and Dip

Next, consider the effects of a sharp increase or decrease. Suppose ix (dip) occurs in a short time. Then, for an individual who values the absolute level, vi and ix give similar lifetime satisfactions while vii and viii give lower ones. For an individual who values the change, because of diminishing marginal utility of change, ix is worse than vi. vii is better than vi, contrary to the absolute-level lovers.

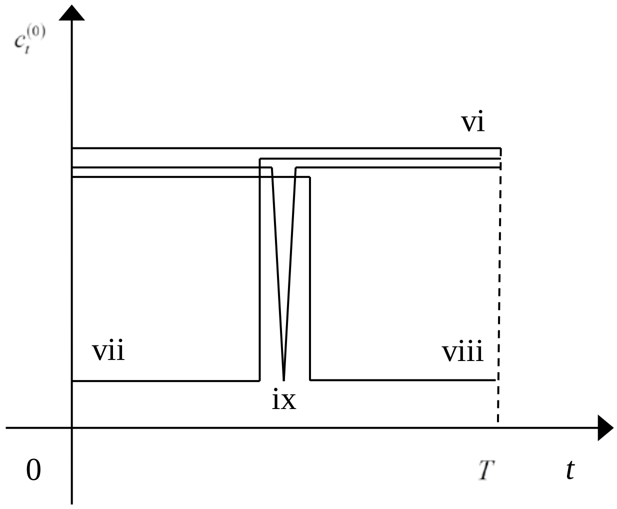

Figure 6.2 Soar, Plunge and Dip

$\alpha_0 > 0$ and $\alpha_1 = \alpha_2 = 0$:

$U^*_{vi} \cong U^*_{ix} > U^*_{vii} = U^*_{viii}$

$\alpha_0 = \alpha_2 = 0$ and $\alpha_1 > 0$:

$U^*_{vii} > U^*_{vi} > U^*_{ix} > U^*_{viii}$

6.3 Second-Order or Rate of the Change

Finally, contrast *increase* with *growth*. An individual who values the absolute level or first-order change prefers a constant increase over a convex one. As has been argued in Sections 3 and 4, an individual who is concerned about the second-order change or who cares for the "growth rate" prefers a convex path. This type of convex path can be also preferred by an individual who takes into account the level of the remaining asset, such as the case in Section 5.

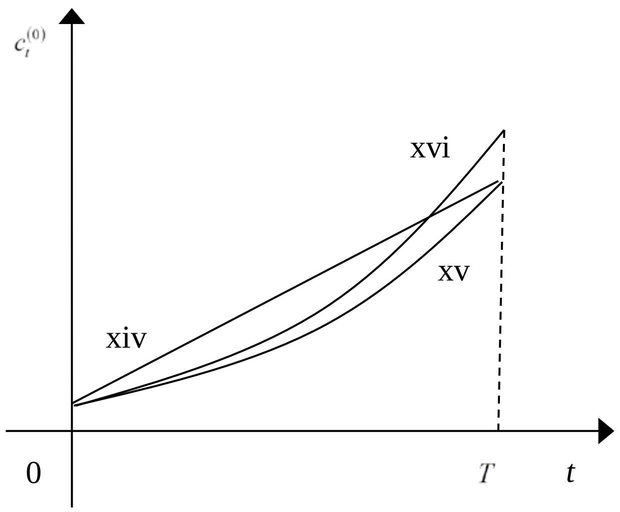

Figure 6.3 Convex Changes

$\alpha_0 > 0$ and $\alpha_1 = \alpha_2 = 0$:

$U^*_{xiv} > U^*_{xvi} > U^*_{xv}$

$\alpha_0 = \alpha_2 = 0$, $\alpha_1 > 0$:

$U^*_{xiv} > U^*_{xvi} > U^*_{xv}$ or $U^*_{xvi} \geq U^*_{xiv} > U^*_{xv}$

$\alpha_0 = 0$, $\alpha_1, \alpha_2 > 0$ or "rate of the change" matters:

$U^*_{xvi} > U^*_{xv} > U^*_{xiv}$

Conclusion

This study has extended Yonemoto (2013) and characterized the behavior of an individual whose instantaneous utility is a function of the (first-order and higher-order) changes in asset as well as consumption. In particular, it has investigated the cases in which an individual takes into account the second-order change in consumption, the rate of the first-order change or the level of remaining asset. It has been shown that, in any case, an individual tends to postpone consumption in earlier periods in comparison with the case in which they do not matter.

In addition, the lifetime utility levels that correspond to several given consumption paths have been compared with each other. Some examples of the reversal in the order, depending on the functional form, have been presented.

Future extensions may include introduction of more general time discount factor, consideration of the effects of the comparison with others (e.g. demonstration effect) and conducting empirical studies which support the theoretical conclusions as well as the assumptions.

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Appendix

The Hamiltonian is:

$$H = u\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right) + \sum_{n=1}^{N} \lambda_t^{c(n)} c_t^{(n)} + \sum_{m=1}^{M} \lambda_t^{a(m)} a_t^{(m)}$$

The Lagrangian is:

$$L = H + \mu_t\, g\left(t, c_t^{(0)}, c_t^{(1)}, \cdots, c_t^{(N)}, a_t^{(0)}, a_t^{(1)}, \cdots, a_t^{(M)}\right)$$

The necessary conditions for maximization are:

i) $$\frac{dc_t^{(n-1)*}}{dt} = \frac{\partial L^*}{\partial \lambda_t^{c(n)*}} = c_t^{(n)*} \quad \text{for} \quad t = 1, \cdots, T \quad \text{and} \quad n = 1, \cdots, N,$$

$$\frac{da_t^{(m-1)*}}{dt} = \frac{\partial L^*}{\partial \lambda_t^{a(m)*}} = a_t^{(m)*} \quad \text{for} \quad m = 1, \cdots, M,$$

ii) $$\frac{d\lambda_t^{c(n)*}}{dt} = -\frac{\partial L^*}{\partial c_t^{(n-1)}} = -\frac{\partial u^*}{\partial c_t^{(n-1)}} - \lambda_t^{c(n-1)*} - \mu_t \frac{\partial g^*}{\partial c_t^{(n-1)}} \quad \text{for} \quad n = 2, \cdots, N,$$

$$\frac{d\lambda_t^{c(1)*}}{dt} = -\frac{\partial L^*}{\partial c_t^{(0)}} = -\frac{\partial u^*}{\partial c_t^{(0)}} - \mu_t \frac{\partial g^*}{\partial c_t^{(0)}},$$

$$\frac{d\lambda_t^{a(m)*}}{dt} = -\frac{\partial L^*}{\partial a_t^{(m-1)}} = -\frac{\partial u^*}{\partial a_t^{(m-1)}} - \lambda_t^{a(m-1)*} - \mu_t \frac{\partial g^*}{\partial a_t^{(m-1)}} \quad \text{for} \quad n = 2, \cdots, M,$$

$$\frac{d\lambda_t^{a(1)*}}{dt} = -\frac{\partial L^*}{\partial a_t^{(0)}} = -\frac{\partial u^*}{\partial a_t^{(0)}} - \mu_t \frac{\partial g^*}{\partial a_t^{(0)}} \quad \text{(adjoint equations,)}$$

iii) $$H\left(t, c_t^{(0)*}, \cdots, c_t^{(N)*}, a_t^{(0)*}, \cdots, a_t^{(M)*}, \lambda_t^{c(1)*}, \cdots, \lambda_t^{c(N)*}, \lambda_t^{a(1)*}, \cdots, \lambda_t^{a(M)*}, \mu_t^*\right)$$

$$= \max_{c_t^{(N)}} H\left(t, c_t^{(0)*}, \cdots, c_t^{(N-1)*}, c_t^{(N)}, a_t^{(0)*}, \cdots, a_t^{(M)*}, \lambda_t^{c(1)*}, \cdots, \lambda_t^{c(N)*}, \lambda_t^{a(1)*}, \cdots, \lambda_t^{a(M)*}, \mu_t^*\right)$$

iv) $$\frac{\partial L^*}{\partial c_t^{(N)}} = \frac{\partial u^*}{\partial c_t^{(N)}} + \lambda_t^{c(N)*} + \mu_t \frac{\partial g^*}{\partial c_t^{(N)}} = 0,$$

v) $\mu_t^* \geq 0$ and $\mu_t^* g\left(t, c_t^{(0)*}, \cdots, c_t^{(N)*}, a_t^{(0)*}, \cdots, a_t^{(M)*}\right) = 0,$

vi) $\frac{dL^*}{dt} = \frac{\partial L^*}{\partial t}$ and

vii) $\lambda_T^{a(0)*} = \frac{\partial \Phi\left(a_T^{(0)*}\right)}{\partial a_T^{(0)}}$ (transversality condition.)

Suppose the constraint is specified by (15) and (16). Then, Euler's equation is derived to be,

$$\sum_{m=0}^{M}\left\{(-1)^m \frac{d^m}{dt^m}\left(\frac{\partial u^*}{\partial a_t^{(m)}}\right)\right\} - \frac{g_{a0}}{g_{c0}}\sum_{n=0}^{N}\left\{(-1)^n \frac{d^n}{dt^n}\left(\frac{\partial u^*}{\partial c_t^{(n)}}\right)\right\}$$

$$+\frac{g_{a1}}{g_{c0}}\sum_{n=0}^{N}\left\{(-1)^n \frac{d^{n+1}}{dt^{n+1}}\left(\frac{\partial u^*}{\partial c_t^{(n)}}\right)\right\} = 0 .$$

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