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Mixed oligopoly in a two-dimensional city

Takanori Ago

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Abstract

This paper analyzes a mixed oligopoly model with one public firm and two private firms in a two-dimensional square city. Three types of spatial equilibria are presented: (i, central symmetry) the public firm locates at the center, the private firms locate equidistantly away from the center on a diagonal in the opposite direction each other; (ii, full agglomeration) the public firm locates slightly away from the center on a diagonal and two private firms agglomerate on the opposite half of the diagonal; (iii, partial agglomeration) in one dimension the locations of private firms coincide but differ from that of the public firm, and with regard to the other dimension the public firm locates at the center and the private firms locate equidistantly away from the center in the opposite direction each other.

Keywords: mixed oligopoly, two-dimensional city, spatial competition

JEL Classification: L13; L15; R39

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[†]Very preliminary. Please do not cite this version without my approval.

[‡]Faculty of Regional Policy, Takasaki City University of Economics, 1300 Kaminamie, Takasaki, Gumma 370-0801, Japan. E-mail: ago@tcue.ac.jp

1 Introduction

Hotelling's (1929) seminal work showed that duopolistic firms agglomerate in the center of a one-dimensional space (a linear city) although d'Aspremont *et al.* (1979) revised the result as *maximum differentiation* under spatial Bertrand competition. On the other hand, Hamilton *et al.* (1989) and Anderson and Neven (1991) developed location-then-quantity (Cournot) competition models. By contrast, they consequently showed the agglomeration of firms in the center (*minimum differentiation*).

Matsushima and Matsumura (2003, henceforth "MM") focused on an observation that in a mixed market private firms often provide a different good or service from public sector but it is similar to other private firms' (herd behavior of private firms). For this to be explained, MM incorporated a welfare-maximizing public firm into a spatial Cournot model. As a result, in their circular city all private firms agglomerate at a point that is the farthest from the public firm. And as an extension, in their linear city there are two types of equilibrium: (i) All private firms agglomerate (Proposition 1, p. 71). (ii) Firms agglomerate at two points if and only if the number of the private firms is even (Proposition 4, p. 73). *These results indicate that public firm has a strong repelling-effect against private firms* because the public firm sets the price of each market at its transport cost (marginal cost pricing), which means markets near the public firm are very competitive for the private firms. At the same time, private firms share the same objective to rival the public firm; they consequently choose the same location.

This paper extends MM into a two-dimensional square city. When it comes to two-dimensional Cournot competition without a public firm, both Berenguer-Maldonado *et al.* (2005) and Ago (2008) have shown that firms agglomerate in the center when space is a disk or a rectangle. Because this is an essentially identical result as in linear space, one may think that we do not need any change in dimensionality. However, with regard to spatial Bertrand competition, dimensionality matters. In their location-then-price competition in a multi-dimensional city, Tabuchi (1994) and Irmen and Thisse (1998) showed that maximum differentiation occurs in only one dimension, while minimum differentiation is achieved in the rest of the dimensions. Hence, with dispersion force or repelling effect, it is important to have a formal result of how firms run away from their rivals, a problem that is more complex and non-trivial because there are many directions for dispersion. That is why this paper tackles on the dimensionality problem.

Another reason for multi-dimension is relevance to the real world. MM suggested "*Television programs supplied by the Japan Broadcasting Corporation (NHK) are quite different from those of private broadcasting companies,* *which are quite similar to each other* (p. 63)." It seems true that a public broadcasting company prefers a more serious program like news to an entertainment or a shopping one, which rather concerns private broadcasting companies. However, with regard to a political aspect, the public company (is effectively required to) chooses a moderate course, while positions of the private companies differ: conservative or liberal stance. Such a complex position can be analyzed by a multi-dimensional model. That is another merit of the paper.

The remainder of the paper is organized as follows. In Section 2, the two-stage location-then-quantity game is presented. In Section 3, quantity choice is analyzed in the second stage by backward induction. Then, Section 4 yields location equilibria in the first stage. Section 5 summarizes the results.

2 The model

Our model is based on a one-dimensional model developed by MM (Matsushima and Matsumura, 2003), who analyzed a location-then-quantity competition model with a welfare-maximizing public firm among private firms. I extend it to a two-dimensional model with simplification in two ways: First, the functional form of transport costs is assumed quadratic in Euclidean distance. Second, the number of the private firms is two. Except for those aspects, MM and this model are identical in order to be better compared.

We consider a two-dimensional city expressed by a square on x-y coordinates, $L = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}$, and consumers are uniformly and continuously distributed on L with a density of one at each location. There are a welfare-maximizing public firm (firm 0) and two profit-maximizing firms (firm 1 and firm 2) that supply a homogeneous good with zero marginal cost. Let firm i's location be $(x_i, y_i) \in L$.

We analyze a two-stage location-then-quantity game with subgame perfection being the equilibrium concept. In the first stage, each firm simultaneously chooses its location. In the second stage, each firm simultaneously chooses its quantity given their locations. Based on the literature, firms bear transport costs and they can set a supply amount for each location independently because arbitrage between consumers is assumed to be prohibitively costly.

Each consumer at (x, y) has the same inverse demand function as follows:

$$P(x, y) = a - bQ(x, y), \quad Q(x, y) = \sum_{i=0}^{2} q_i(x, y), \tag{1}$$

where $P(x, y)$ is the price at (x, y), $q_i(x, y)$ $(i = 0, 1, 2)$ and $Q(x, y)$ is supply amount of each firm and the total supply amount there. a and b are positive constants.

The transport costs are the same for the firms and are linear to supply amount. The unit transport cost is quadratic with regard to Euclidean distance between a firm and a consumer. Let $d_i(x, y)$ denote the distance between a consumer at (x, y) and firm i. Then, the transport cost is given by

$$td_i(x,y)^2 = t[(x_i - x)^2 + (y_i - y)^2],$$

where t is the transport cost parameter and is assumed to be sufficiently low such that

$$a > 24t, \tag{2}$$

which ensures that the public firm always serves the entire city, irrespective of the locations of the firms.[1]

For private firm i, the local profit at (x, y) and the total profit are given

[1] More generally, the condition is $a > (n+1)\bar{t}$, where n is the number of private firms and $\bar{t}$ is the maximal transport cost for the public firm. In our model, $n = 2$ and $\bar{t} = t\,(\max d_0(x,y))^2 = t\left(\sqrt{2^2+2^2}\right)^2$ yield the threshold of (2).

by

$$\begin{aligned}\pi_i(x,y) &= q_i(x,y)\left[P(x,y) - td_i(x,y)^2\right],\\ \Pi_i &= \iint_L \pi_i(x,y)dxdy,\end{aligned}$$

where $P(x,y)$ is given by (1). Let $w(x,y)$ denote the social surplus (consumer surplus plus the sum of the profits) at (x,y):

$$w(x,y) = \int_0^{Q(x,y)} (a - bm)\, dm - \sum_{i=0}^{2} td_i(x,y)^2 q_i(x,y),$$

and the (total) social surplus is

$$ss = \iint_L w(x,y)dxdy.$$

3 Quantity choice

First, we analyze the second-stage game. In this stage, we can apply the same analysis in MM because only distance matters, irrespective of dimensionality. Therefore, following important results in MM are clearly valid here as well:

Remark 1 *(Lemma 1 in MM, p.69) In equilibrium,*

$$P(x,y) = td_0(x,y)^2,\ Q(x,y) = \frac{a - td_0(x,y)^2}{b}. \tag{3}$$

Remark 2 *(Lemma 2 in MM, p.69) Consumer surplus, cs, does not depend on (x_i, y_i) $(i = 1, 2)$ and is given by*

$$\begin{aligned} cs &= \iint_L cs(x,y)dxdy \\ &= \frac{2}{45b}[45a^2 - 30at\left(2 + 3x_0^2 + 3y_0^2\right) \\ &\quad +t^2\left(28 + 45x_0^4 + 45y_0^4 + 120x_0^2 + 120y_0^2 + 90x_0^2y_0^2\right)], \end{aligned} \tag{4}$$

where $cs(x,y)$ is consumer surplus at (x,y), which is

$$cs(x,y) = \frac{\left[a - td_0(x,y)^2\right]^2}{2b}.$$

Note that if we rewrite (x_0, y_0) as $(r\cos\theta, r\sin\theta)$, then we have

$$cs = \frac{2}{45b}\left[45a^2 - 30at\left(2 + 3r^2\right) + t^2\left(28 + 120r^2 + 45r^4\right)\right],$$

and

$$\frac{\partial cs}{\partial \theta} = 0,\ \frac{\partial cs}{\partial r} = 60tr\left[-3a + \left(4 + 3r^2\right)t\right] < 0,$$

where the inequality is due to (2). Therefore, the nearer to the center the public firm is, the greater cs becomes.

Remark 3 *(Lemmas 3 and 4 in MM, p.70) Firm i $(i = 1, 2)$ supply positive amount of $q_i(x, y)$ at (x, y) if and only if $d_0(x, y) > d_i(x, y)$, where*

$$q_i(x, y) = \frac{t\left[d_0(x, y)^2 - d_i(x, y)^2\right]}{b},$$

and its profit is

$$\pi_i(x, y) = b\left[q_i(x, y)\right]^2.$$

Remark 4 *(Lemma 5 in MM, p.71) The profit of each private firm does not depend on the location of the other private firm:*

$$\Pi_i(x_i, y_i; x_0, y_0) = \iint_{L_i} \pi_i(x, y)dxdy, \tag{5}$$

where L_i $(i = 1, 2)$ denotes the domain in which firm i serves:

$$L_i = \{(x, y) : d_0(x, y) > d_i(x, y)\} \cap L.$$

Note that the market boundary of $d_0(x, y) = d_i(x, y)$ is the perpendicular bisector of the line segment jointing the locations of firm 0 and firm i.

4 Location equilibrium

We analyze the first-stage game given the results in the second stage. Without loss of generality, we henceforth assume that $0 \leq x_0, y_0 \leq 1$. Furthermore, due to symmetry and for notational convenience, we will omit any equilibria that is symmetric with another equilibrium: For example, if $\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\} = \{(0,0), (1/2, 1/2), (-1,-1)\}$ were an equilibrium; $\{(0,0), (-1/2, -1/2), (1,1)\}$, which were clearly another equilibrium, would be omitted because it is regarded as essentially the same equilibrium.

First of all, symmetry yields the following intuitive result.

Lemma 1 *The public firm locates on x-axis (y-axis) if and only if the private firms locate equidistantly away from x-axis (y-axis) in opposite direction each other. In other words, in equilibrium, $x_0 = 0 \iff x_1 + x_2 = 0$ and $y_0 = 0 \iff y_1 + y_2 = 0$.*

Proof. See the appendix. ■

This lemma classifies the types of location equilibria. For the public firm to choose the center, $(x_0, y_0) = (0,0)$, the private firms must locate symmetrically with regard to the center, $(x_2, y_2) = (-x_1, -y_1)$. Let this case be named *central symmetry*, which will be analyzed in Subsection 4.1 below.

On the other hand, unless the public firm locates on either of two axes, symmetry breaks down. From Remark 4, the profit functions of both private firms are only dependent on locations of public firm and of its own; hence, both firms would choose the same location, $(x_1, y_1) = (x_2, y_2)$. We name this case *full agglomeration*, which we will deal with in Subsection 4.2.

At last, there is another possibility: the public firm locate on one axis but not on the other axis. For example, suppose that $x_0 = 0$ and $y_0 \neq 0$. In this case, $x_2 = -x_1$ must hold, but the symmetry with regard to y-coordinate has broken down. Again from Remark 4, the private firms would locate the same position with regard to y-coordinate. We call it *partial agglomeration*. Subsection 4.3 will see the case.

Among three cases above, central symmetry and agglomeration were already presented in the linear-city case of MM. Meanwhile, the last case is new.

4.1 Central symmetry

First, we focus on the possibility that the public firm is located at the center, $(x_0, y_0) = (0, 0)$ and the private firms are symmetrically located with regard to the public firm in equilibrium.

Proposition 1 *Under the scheme of central symmetry, there exists a unique* *location equilibrium*[2] *such that*

$$(x_0, y_0, x_1, y_1, x_2, y_2) = (0, 0, -2/3, -2/3, 2/3, 2/3).$$

Proof. See the appendix. ■

This equilibrium corresponds to the equilibrium in MM (Proposition 4 (ii) in MM), where the public firm locates at 1/2, half of the private firms locate at 1/10 and the others locate at 9/10 in a linear city of a line segment $[0, 1]$[3].

Unlike Tabuchi (1994) and Irmen and Thisse (1998), private firms differentiate with regard to all dimensions here. Irmen and Thisse (1998) give us a clue how to interpret an equilibrium in multi-dimensional models:

> In a symmetric equilibrium with fixed market size profits are highest when prices are highest, that is, when the elasticity of demand is lowest. The lower the density of marginal consumers, the lower is the elasticity. Accordingly, as the consumer distribution is uniform, demand has minimal elasticity when the corresponding hyperplane has minimal surface area. Since the

[2] *Remember the sentence at the top of this section. We omit other symmetric equilibria like* $(x_0, y_0, x_1, y_1, x_2, y_2) = (0, 0, 2/3, -2/3, -2/3, 2/3)$ *because this is essentially the same equilibrium in this proposition.*

[3] If we change the space into a rectangle, $\{(x, y) \in \mathbb{R}^2 : -c \leq x \leq c, -1/c \leq y \leq 1/c\}(c > 1)$; we can find when c gets large, the equilibrium approaches MM's.

strategy space is a hypercube this hyperplane is the one that is parallel to one of the facets, which in turn implies that the products differ only in one dimension.

In this paper, when $(x_0, y_0, x_1, y_1) = (0, 0, -2/3, -2/3)$ in Proposition 1, the length of market boundary between firm 0 and firm 1 is $1.88562 = 4\sqrt{2}/3$. On the other hand, if $(x_0, y_0, x_1, y_1) = (0, 0, -1, 0)$, which gives differentiation in one dimension; the length of market boundary is 2. Therefore, the elasticity of demand is lower in differentiation in both dimensions.

Needless to say, locating at $(x_1, y_1) = (-1, -1)$ minimizes the boundary when $(x_0, y_0) = (0, 0)$. However, Cournot competition has a relatively stronger centripetal force (Hamilton *et al.,* 1989; Anderson and Neven, 1991). Therefore, the tradeoff between these effects yields our interior location equilibrium.

4.2 Full agglomeration

Next, we analyze the possibility that the private firms agglomerate. Unfortunately, due to complexity (high order of equations), it is impossible to get the solutions analytically. Hence, we will investigate some properties of equilibrium and conduct a numerical analysis below.

At first, remember that cs depends on a ($\partial cs/\partial a > 0$) but the profits of the private firms do not (Remark 2 and Remark 4). Hence, when a is sufficiently large, the public firm cares the consumer surplus much more. Clearly, when $a \to \infty$, the public firm chooses $(x_0, y_0) = (0, 0)$. When $(x_0, y_0) = (0, 0)$, as Subsection 4.1 shows, a private firm would choose $(x_1, y_1) = (-2/3, -2/3)$. Because of independence from the other firm's location on its profit, we have the following.

Proposition 2 *Under the scheme of full agglomeration, we have*

$$\lim_{a \to \infty} (x_0, y_0, x_1, y_1, x_2, y_2) = (0, 0, -2/3, -2/3, -2/3, -2/3).$$

However, from Lemma 1, $(x_0, y_0) = (0, 0)$ is consistent only when the case of central symmetry. Therefore, when a is very large but not infinity, the public firm would locate not at the center but near the center due to continuity. Then, we proceed to an analysis of the first-order conditions, $\partial ss/\partial x_0 = 0$, $\partial ss/\partial y_0 = 0$, $\partial \Pi_i/\partial x_i = 0$ and $\partial \Pi_i/\partial y_i = 0$ $(i = 1, 2)$. Let

$$0 < \tau \equiv t/a < \frac{1}{24}, \tag{6}$$

then the solutions depend only on τ. In the vicinity of $(x_0, y_0, x_1, y_1, x_2, y_2) =$

$(0, 0, -2/3, -2/3, -2/3, -2/3)$, a comparative statics yields

$$\frac{dx_0}{d\tau} = \frac{dy_0}{d\tau} = 3\frac{dx_i}{d\tau} = 3\frac{dy_i}{d\tau} = \frac{512}{729 - 2252\tau} > 0 \ (i = 1, 2),$$

where the inequality is due to (6). This result indicates that the public firm locates slightly away from the center to a point on a diagonal of L while the private firms slightly approach the public firm on the same diagonal when τ is sufficiently small (a is very large).

Unfortunately, all we can do is to proceed to a numerical analysis as a further investigation because of analytical unsolvability. The analysis yields the following numerical result.

Solution 3 *Under the scheme of full agglomeration, the public firm locates near the center on a diagonal while the private firms agglomerate at a point on the opposite half of the diagonal with regard to the public firm. When τ increases from zero, the public firm goes away from the center; while the private firms approach the center from the point that divides internally the half of the diagonal in the ratio of 2:1.*

Figure 1 and Figure 2 around here.

Figure 1 and Figure 2 show this result, which corresponds to an equilibrium of Proposition 4 (i) in MM. This result is consistent with our intuition. The greater τ becomes, the more the public firm cares the profits of the private firms. Suppose that all firms are on a diagonal, then the movement along the diagonal is the best way for separating from the private firms with the loss of the consumer surplus being minimized because the consumer surplus is dependent only on the distance between the public firm and the center. Then, once the public firm goes away from the center, the private firms have an incentive to approach the center due to relaxed competition.

4.3 Partial agglomeration

We can guess from Lemma 1 that if the public firm locates on either x-axis or y-axis, then there is an equilibrium where the private firms are symmetric with regard to that axis. Unfortunately, because the analytical unsolvability continues, we will repeat a similar analysis as in the case of full agglomeration. Without loss of generality, we only deal with the case where the public firm locates on y-axis ($x_0 = 0$).

First of all, the limit case of $a \to \infty$ yields the same outcome as in the case of full agglomeration (Proposition 2).

Proposition 4 *Under the scheme of partial agglomeration, we have*

$$\lim_{a\to\infty}(x_0, y_0, x_1, y_1, x_2, y_2) = (0, 0, -2/3, -2/3, 2/3, -2/3).$$

Next, we proceed to an analysis in the vicinity of the limit case above. Letting $x_0 = 0$, a comparative statics that is evaluated at $(y_0, x_1, y_1, x_2, y_2) = (0, -2/3, -2/3, 2/3, -2/3)$ yields

$$\begin{aligned}
\frac{dy_0}{d\tau} &= \frac{512}{729 - 1996\tau} > 0, \\
\frac{dx_1}{d\tau} &= -\frac{dx_2}{d\tau} = \frac{1024}{2187 - 5998\tau} > 0, \\
\frac{dy_1}{d\tau} &= \frac{dy_2}{d\tau} = -\frac{512}{2187 - 5998\tau} < 0,
\end{aligned}$$

where the inequalities are due to (6). Note that

$$\frac{dy_0}{d\tau} : \frac{dx_1}{d\tau} : \frac{dy_1}{d\tau} = 1 : \frac{2}{3} : -\frac{1}{3}.$$

Hence, when the public firm moves slightly away from the center in the direction of north, firm 1 moves in the direction of southeast from $(x_1, y_1) = (-2/3, -2/3)$. A numerical analysis confirms such a successive movement as follows.

Solution 5 *Suppose that the public firm is on y-axis ($x_0 = 0$). When τ*

increases, the public firm goes away from the center on y-axis while firm 1 *goes southeastern away from* $(-2/3, -2/3)$ *with firm* 2 *being kept symmetric with regard to y-axis.*

Figures 3-5 around here.

Figures 3-5 show this result. This solution seems somewhat different from that in the case of full agglomeration, where the public firm exhibits a strong repelling effect: When the public firm approaches the center, the private firms move away from it. In the case of partial agglomeration, the private firms do move away in one dimension although they approach the public firm in the other dimension. The market boundary is a key to understand.

Suppose that $(x_1, y_1) = (-2/3, -2/3)$. If the public firm slightly moves northern from the center $(x_0, y_0) = (0, \varepsilon)$, the boundary rotates counterclockwise a little bit. Hence, the northwestern markets become more competitive while the southeastern markets become less competitive. Therefore, firm 1 has an incentive to move southeastern, by which the firm can avoid keen competition.

4.4 Welfare implications

We compute the consumer surplus, the profits, and the social surplus in equilibrium. Note that only consumer surplus depends on a; the greater a becomes, the greater the consumer surplus. Meanwhile, b is just a multiplier for each value. Let CS_j, PS_j and SS_j $(j = sym, agg, par)$ denote consumer surplus, producer surplus (profits) and social surplus, respectively; where sym, agg, par denote the three cases (central symmetry, full agglomeration, partial agglomeration in order). Then, we have the following result.

Solution 6 *$CS_{\boldsymbol{agg}} < CS_{par} < CS_{sym}$, $PS_{agg} > PS_{par} > PS_{sym}$ for each t/a. The difference between each pair is increasing in t. $SS_{agg} \lesseqgtr SS_{par} \lesseqgtr SS_{sym}$ when $a \gtreqless \tilde{a}(t)$, where $\tilde{a}(t)$ is a threshold of a function of t.*

These orders are quite intuitive. Because the consumer surplus is maximized when the public firm locates at the center, central symmetry makes the consumer surplus maximal. On the other hand, roughly speaking, because the private firms' profits increase when the public firm goes away from the center, central symmetry minimizes the producer surplus. Hence, social surplus depends on which surplus is more important. If a is sufficiently large, consumer surplus is dominant. Therefore, central symmetry maxmizes the social surplus when a is sufficiently large.

5 Conclusion

We have analyzed an extended model of spatial mixed oligopoly. We have had some similar results as in a linear city of MM (Proposition 1 and Solution 3). As a new result, we have Solution 5, where the private firms differentiate only in one dimension. Further, the reactions of the firms to a change of transport costs are analyzed. In our two-dimensional model, when the public firm approaches a private firm, the private firm can more differentiate in a dimension while it less differentiates in the other.

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APPENDIX

Proof of Lemma 1

Proof. Differentiating the social surplus with regard to x_0 and evaluating it at $x_0 = 0$, we have

$$\left.\frac{\partial ss}{\partial x_0}\right|_{x_0=0} = -\frac{32t^2}{3b}\left(x_1 + x_2\right).$$

The first-order condition requires $x_1 + x_2 = 0$. Furthermore, using (2), we can readily confirm $\partial^2 ss/\partial x_0^2 < 0$. Therefore, for $x_0 = 0$ to be an equilibrium, $x_1 + x_2 = 0$ is required.

Conversely, assume that $x_1 + x_2 = 0$. Then, we have

$$\left.\frac{\partial ss}{\partial x_0}\right|_{x_2=-x_1} = -\frac{8t^2}{b}\left(a/t + 4x_1^2 + 2y_1^2 + 2y_2^2 - 5x_0^2 - 5y_0^2 - 4\right)x_0.$$

Because $a/t > 24$ from (2), the value of the parenthesis is positive. Therefore, ss is maximized at $x_0 = 0$.

Hence, the public firm locates at $x_0 = 0$ if and only if $x_1 + x_2 = 0$. Due to symmetry, we can apply identical analysis with regard to y-coodinate. ■

Proof of Proposition 1

Proof. Without loss of generality we assume that $-1 \leq y_1 \leq x_1 \leq 0$ with $(x_2, y_2) = (-x_1, -y_1)$. First, when $(x_1 - 1)^2 + (y_1 + 1)^2 \geq 2$, the boundary of $d_0(x, y) = d_1(x, y)$ intersects the lines of $x = -1$ and $y = -1$. Hence, firm 1's profit is given by

$$\Pi_1 = \frac{t^2 \left[(x_0 + 1)^2 + (y_0 + 1)^2 - (x_1 + 1)^2 - (y_1 + 1)^2\right]^4}{48b\,(x_0 - x_1)\,(y_0 - y_1)}$$

and due to symmetry, firm 2's profit is

$$\Pi_2 = \frac{t^2 \left[(x_0 + 1)^2 + (y_0 + 1)^2 - (-x_1 + 1)^2 - (-y_1 + 1)^2\right]^4}{48b\,(x_0 + x_1)\,(y_0 + y_1)}.$$

Then, the social surplus is given by

$$ss = cs + \Pi_1 + \Pi_2,$$

where cs is given by (4). We can readily show that ss is maximized at $(x_0, y_0) = (0, 0)$ from the first-order and the second-order conditions for the public firm. Next, the first-order conditions for firm 1 evaluating at

$(x_0, y_0) = (0, 0)$ and $(x_2, y_2) = (-x_1, -y_1)$ yield the following equations:

$$6x_1 + 7x_1^2 - 2y_1 - y_1^2 = 0, 6y_1 + 7y_1^2 - 2x_1 - x_1^2 = 0.$$

The admissible solution is only $(x_1, y_1) = (2/3, 2/3)$, which satisfies the second-order condition.

Second, when $(x_1 - 1)^2 + (y_1 + 1)^2 < 2$, the boundary of $d_0(x, y) = d_1(x, y)$ intersects the lines of $x = 1$ and $x = -1$. Similar calculation yields the social surplus and the profits, and we can show that no interior solutions exist for profit maximization and the profit functions are differentiable at eah point on $d_0(x, y) = d_1(x, y)$. Therefore, the location pair of $(x_0, y_0) = (0, 0), (x_1, y_1) = (-2/3, -2/3), (x_2, y_2) = (2/3, 2/3)$ is a unique equilibrium when $-1 \leq y_1 \leq x_1 \leq 0$. Clearly, due to symmetry, we have the result. ■

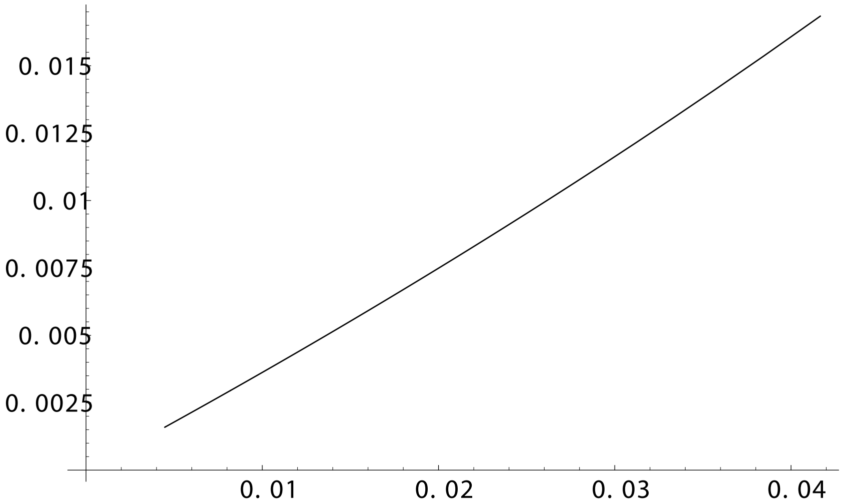

Figure 1: The location of the public firm. (The horizontal axis shows t/a and the vertical axis shows x0(= y0).)

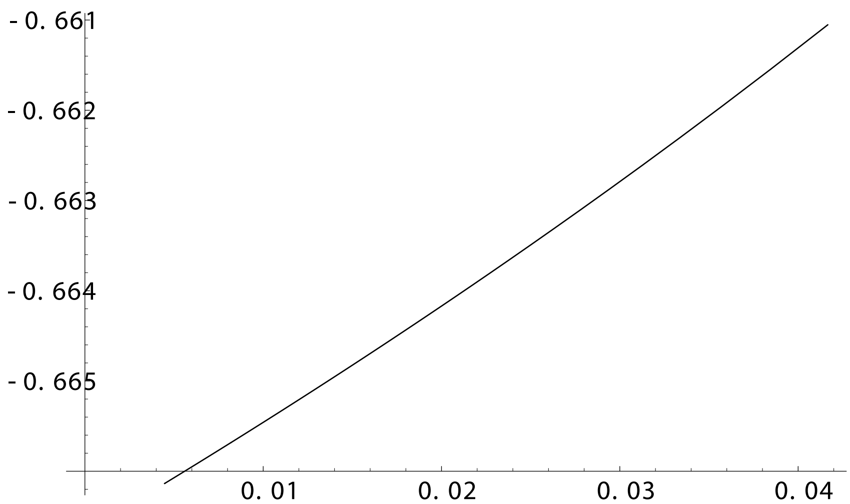

Figure 2: The location of the private firm. (The horizontal axis shows t/a and the vertical axis shows x1(= y1).)

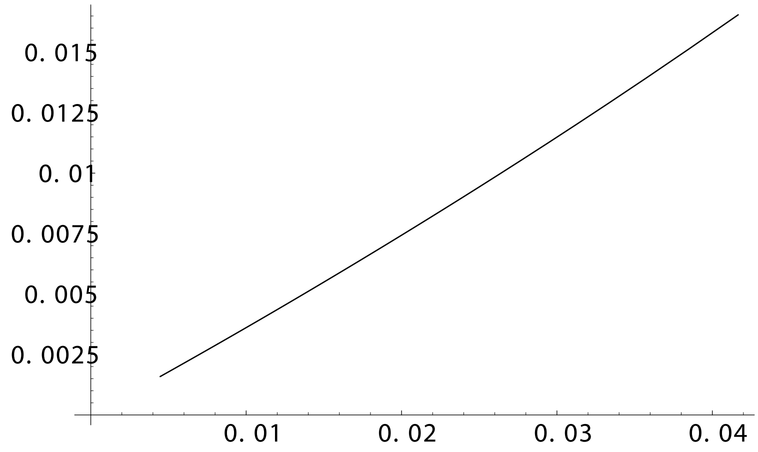

Figure 3: The location of the public firm given x0 = 0. (The horizontal axis shows t/a and the vertical axis shows y0.)

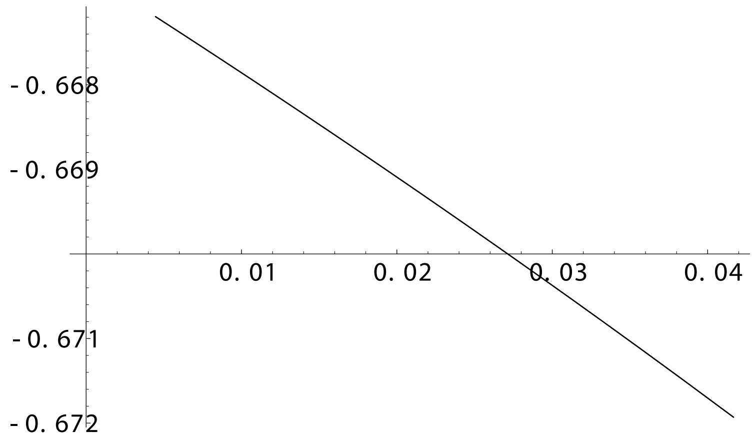

Figure 4: The location of the private firm. (The horizontal axis shows t/a and the vertical axis shows y1.)

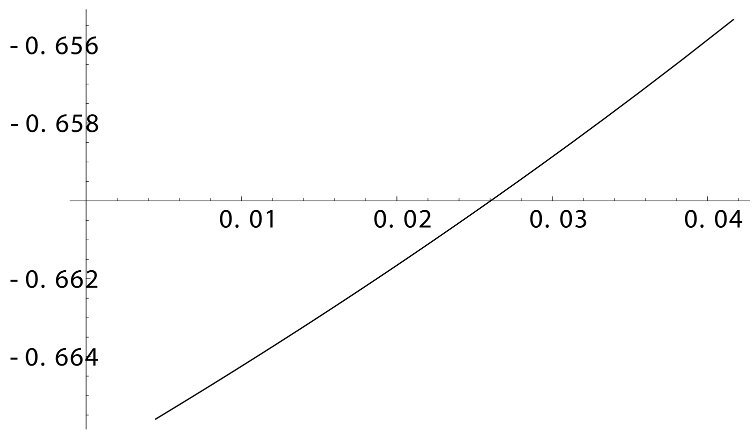

Figure 5: The location of the private firm. (The horizontal axis shows t/a and the vertical axis shows x1.)

高崎経済大学地域政策学会
370-0801 群馬県高崎市上並榎町１３００
027-344-6244
c-gakkai@tcue.ac.jp
http://www1.tcue.ac.jp/home1/c-gakkai/dp/dp09-02